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CONTENTS

THE EDITOR'S PAGE

MATHEMATICAL MACHINES

Two news items have come to our attention recently concerning the wonders of our modern mathematical world. Did you know that an electronic computer has recently been used to prepare and edit material for the printing of a book, A CONCORDANCE TO THE POEMS OF MATTHEW ARNOLD ? Cornell University Press revealed that they published the 965 page tome containing 10,097 words of Arnold's vocabulary and some 70,000 references. The work was done by using an IBM 704 Data Processing System. Nothing is safe from mathematics these days.

The National Bureau of Standards has been investigating a possible mathematical wedding. They are contemplating a machine that is both a digital calculator and an analog computer. Their proposed analog-digital differential analyzer will combine the analog advantages of high speed and continuous representation of variables with the digital capability for high precision and dynamic range. Happy honeymoon!

THANKS

The editors wish to express our appreciation to the following who have assisted us by refereeing some of our manuscripts: H. L. Alder, E. F. Beckenbach, Leonard Carlitz, D. O. Ellis, P. J. Kelly, Brockway McMillan, Sam Perlis, H. O. Pollak, W. T. Reid, Richard Scalettar, Olga Taussky Todd.

..., 49, 50.

We are happy to welcome Alaska and Hawaii as our 49th and 50th states by publishing as the two lead articles in this issue the papers by Benjamin L. Schwartz of Hawaii and F. D. Parker of Alaska.

WHAT COLOR WAS THE BEAR?

Benjamin L. Schwartz

Our purpose in this note is two-fold. Firstly, we shall re-examine one of the oldest and best known "chestnuts" in mathematical puzzle literature. It will be discovered that despite its age and the considerable thought which has been devoted to it, the complete solution is generally not known. In fact, to the writer's knowledge, the results presented below have never been published before in their entirety. In carrying out this study, we shall employ some principles of mathematical investigation which are highly respected and universally approved, but often ignored in practice! Our second purpose is to draw attention anew to these principles by pointing out, through the amusing example of the puzzle, the errors which can ensue from bypassing them.

Our point of departure is the problem given below. Although several variants are current, we believe the formulation we give to be typical. The statement begins:

An explorer on the surface of the earth (assumed spherical) sees a bear 100 yards due south of him. The bear then travels 100 yards due east while the explorer remains stationary. The explorer now fires a shot due south, which travels straight and true, and strikes and slays the bear.

At this point, it is customary to ask, "What color was the bear?" The intended answer is, "white", since the sequence of events is supposed to be possible only if the explorer were at the North Pole; hence, the bear must be one of the polar variety.

Without considering whether the answer is correct, we can say that the reasoning in this explanation is definitely incorrect. The North Pole is not the only location at which the conditions can be met, as we shall show. Therefore, rather than the quaint color-of-the-bear formulation, we shall ask more prosaically for all possible locations on the earth where the described events could occur.

It is intuitively obvious that any answer will be independent of longitude, hence all answers will be circles of latitude (in the case of the North Pole solution, a degenerate circle of zero radius). However, one of the points we wish to emphasize is that the use of intuition can be deceptive, and should be avoided. Hence, we shall allow the preceeding result to emerge as a result of our formal analysis rather than impose it *a priori* on the analytic formulation.

Our method will be to employ the familiar spherical coordinate system with origin at the center of the earth. The coordinate θ will be the longitude west of Greenwich (say), and ϕ the colatitude south of the boreal pole. These concepts will be made more precise presently.

Let $P_1 = (R, \theta_1, \phi_1)$ be the explorer's position, where R is the radius of the earth in yards. Then $P_2 = (R, \theta_1, \phi_1 + 100/R)$ will be the initial position of the bear, 100 yards due south. Likewise, the final position of the bear, 100 yards due east of P_2 will be $P_3 = (R, \theta_1 + 100/[R\sin(\phi_1+100/R)], \phi_1 + 100/R)$.

For P_3 to be due south of P_1, we apparently require that $\theta_3 = \theta_1 + 100/[R\sin(\phi_1 + 100/R)]$ be identical with θ_1. Now this is readily shown to be impossible. Wherein lies the difficulty? At least one solution is known to exist, yet the coordinate geometry says there are none!

The vital clue can be found by examining the known solution to determine why its conditions do not agree with the calculations. The answer reveals itself at once. The trouble lies in the coordinate system. It seems clear that a careful re-examination of the spherical coordinate system's peculiarities is in order. Let us review with greater precision how the coordinates are defined. Supposing for convenient reference a Cartesian coordinate system already extant (and bypassing the logical problems this assumption entails), we have the spherical coordinates (r, θ, ϕ) of a point P whose Cartesian coordinates are (x, y, z) given by:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \sin^{-1}\frac{x}{\sqrt{x^2+y^2}} = \cos^{-1}\frac{y}{\sqrt{x^2+y^2}}$$

$$\phi = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$

The multiple definitions of θ are not completely superfluous, since the appropriate value of the multiple-valued inverse trigonometric functions must be selected. In fact, the given relations are sufficient to accomplish this only to within a multiple of 2π, and an additional restriction, such as $0 \leqq 0 < 2\pi$, is needed to complete the definition.

Now numerous potential difficulties appear. At the origin, both θ and ϕ are undefined. Elsewhere along the z-axis, θ is undefined. And along the half plane $y = 0$, $x \geqq 0$, θ is discontinuous.

At this point the reader might wonder whether it would not be easier to give up this approach entirely and start over with another coordinate system. Unfortunately, this would not help. The latitude and longitude are "natural" coordinates of the problem because the directions given are described in terms of points of the compass. With any other coordinate system, we would merely be required to find a way to express southward and eastward travel, i.e., latitude and longitude movement, in terms of other variables.

The difficulties which have plagued us would still occur, since the very concepts of southness and eastness have these characteristics. Expressing them in other coordinates would complicate but not remove the singularities.

To determine the effect of the various special properties of the coordinate system in any particular problem, such as the Explorer-Bear puzzle, we must consider the possibility that the various paths of interest in the problem meet or cross the singular points of the coordinate system. When the paths are limited (as they are in our example) to the surface of the earth, the locus of singularities of the coordinate system turns out to be the Greenwich Meridian $\theta = 0$, from North Pole to South Pole, inclusive, but not including the International Date Line, $\theta = \pi$.

To deal with the possible paths meeting this locus, several alternatives exist. The most natural one, perhaps, is to introduce a Riemann surface type of covering space with an analytic structure induced by θ. The effect of this is to make θ an analytic and univalent function on the Riemann surface. For any point on the Riemann surface, the value of θ will be congruent modulo 2π to true longitude on the corresponding point on the base space sphere.

If the geometry of the puzzle is now expressed in terms of the coordinate system on this Riemann surface, the analytic expressions for P_1, P_2 and P_3 turn out to be identical to those previously developed. The difference is that we now require θ_3 to be *coterminal* with θ_1, i. e., $\theta_3 \equiv \theta_1\ (2\pi)$. This leads to the relation

$$100 = 2\pi k R \sin(\phi_1 + 100/R) \qquad \text{for} \qquad k = 0, \pm 1, \pm 2, \cdots.$$

It is easy to see by examination of the problem statement that zero and negative values of k are unacceptable. Hence it follows that

$$\phi_1 = \sin^{-1}(100/2\pi k R) - 100/R\ , \qquad k = 1, 2, 3, \cdots$$

gives solutions to the puzzle. The appropriate inverse sine is readily determined to be $\pi - \sin^{-1}(100/2\pi k R)$. The solutions here obtained may be described in words as follows. The explorer can be located anywhere on a circle of latitude which is 100 yards north of the circle of latitude whose length (as a small circle) is $100/k$ yards. The bear will execute k complete trips around the South Pole, returning to his starting point before meeting his fate. Clearly these points satisfy the problem statement. However, as we shall see, even these do not exhaust the possible solutions.

The North Pole solution has still not shown up! Even on the Riemann surface, the extended coordinate system still has singularities along the z-axis. Hence, on the surface of the earth, the two poles must still be analyzed separately. Individual examination must be made to determine if the problem can have solutions which meet or pass through either of these singularities.

The eastern path of the bear cannot involve a pole, but either the explorer's original line of sight or his final line of aim might involve transpolar considerations. The North Pole solution falls out at once when we investigate these points. But the South Pole can also enter. The bear is originally south of the explorer. No possibility therefore exists that the pole lies between man and beast.* But the gunshot! It is fired due south, but no requirement of the problem prevents it from passing over the pole and continuing north to strike the bear. Such a situation is, in fact, consistent with the problem wording, and can be achieved if the bear completed $k+1/2$ trips around the pole, for $k = 0, 1, 2, \cdots$ This admits another family of circles of latitude defined by

$$\phi_1 = \pi - \sin^{-1} \frac{100}{2\pi R(k+1/2)} - \frac{100}{R}, \quad k = 0, 1, 2, \cdots.$$

These circles are 100 yards north of circles whose circumferences are $100/(k+1/2)$ yards. These supplement the previously described partial solutions to give the complete answer to this problem.

The last-described loci for the bear, the circles of latitude of circumferences, $66-2/3$ yards, 40 yards, $28-4/7$ yards, etc., do not seem to be generally known, and their presentation here may therefore contribute to the knowledge of some readers.

Two lessons can be inferred from our discussion of this old warhorse of a problem. Both are reasonably well known, but are important enough to stand restatement here. First, generally speaking, mathematical problems (including puzzles) are best solved by rigorous mathematical methods, in preference to intuitive consideration. Recognition of this in the Explorer-Bear example would forestall anyone overlooking the southern hemisphere solutions on the naive assumption that the North Pole solution is unique. The analytic geometry properly interpreted immediately discloses the other solutions. Secondly, when mathematical methods are employed, consideration must be given to their limitations and range of applicability.

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*Note added in proof. It has recently been pointed out to the author that additional solutions can be introduced by a minor change in the problem wording. Let the first sentence be changed as follows: "An explorer on the surface of the earth (assumed spherical) looks due south and sees a bear 100 yards away, directly in his line of sight".

We leave the reader to determine what the new solutions are, and what lesson can be drawn from their existence.

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MATRICES, RELATIONS, AND GRAPHS

F. D. Parker

From its infancy barely a century ago, the theory of matrices has become a sprawling giant, invading every branch of pure and applied mathematics. As late as twenty years ago, few college undergraduates had a knowledge of matrices; today we introduce college freshmen and even high school students to matric methods.

Vast as the literature of matrices is, the field is young enough and varied enough for professional and amateur alike. It might seem that if we restrict our investigation to square matrices whose elements are zero or unity, we might find only a few applications, yet the evidence is to the contrary. Even a superficial investigation answers many questions and at the same time suggests many new questions, which is exactly what a mathematical investigation should do.

This paper treats only a few such applications and probably none of these are completely new.

Communications

Consider a collection of persons (points, telephones, etc.) in which each person either does or does not communicate directly with every other person. An example of six persons is shown in Figure 1. The arrows show

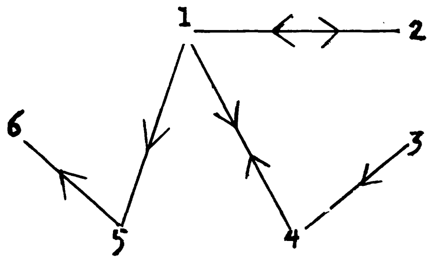

Figure 1.

that the first person communicates with the second, but not the sixth, the third person communicates with the fourth, but the fourth does not communicate with the third, etc. The situation can be neatly described by the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where unity indicates communication, zero lack of communication.

While the matrix representation is no immediate simplification, it can quickly answer such questions as the following: how many persons does the ith person communicate with directly, how many persons communicate directly with the ith person, in how many ways can the ith person communicate with the jth person through one intermediary, can every person (directly or indirectly) communicate with the ith person? These can be answered respectively by finding the sum of the ith row, the sum of the ith column, the element in row i, column j of M^2, and the elements of the ith column of the matrix $M+M^2+M^3+M^4+M^5$.

Some questions immediately are raised which are not answered as quickly. What matrices are equivalent to this matrix, in the sense that the same system of communication (direct or indirect) is maintained? Of these equivalent matrices, which one has the least (greatest) number of lines?

It may be instructive for the reader to verify that one matrix equivalent to M is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Relations

If an element x of a set is related to another element y of the same set, we say that $x\mathrm{R}y$, otherwise $x\overline{\mathrm{R}}y$. We say that the relation is reflexive if $x\mathrm{R}x$ for all x, is symmetric if $x\mathrm{R}y$ implies $y\mathrm{R}x$, transitive if $x\mathrm{R}y$ and $y\mathrm{R}z$ implies $x\mathrm{R}z$, irreflexive if $x\overline{\mathrm{R}}x$, antisymmetric if $x\mathrm{R}y$ implies $y\overline{\mathrm{R}}x$, intransitive if $x\mathrm{R}y$ and $y\mathrm{R}z$ implies $x\overline{\mathrm{R}}z$.

Each relation is characterized by a square matrix with unity in the appropriate place if $x\mathrm{R}y$, otherwise zero. A reflexive relation has unity everywhere in the diagonal terms of the matrix, an irreflexive relation has zeros in the diagonal terms, a symmetric relation has a symmetric matrix, etc. A relation is transitive if and only if whenever an element of M^2 is non-zero, the corresponding element of M is likewise non-zero. If a relation is an equivalence relation (reflexive, symmetric and transitive), the set may be separated into mutually exclusive and collectively exhaustive equivalent subsets. In the language of matrices, the inter-change of two columns followed by the interchange of the same two rows, followed by two similar interchanges, etc. will eventually diagonalize the matrix, showing the equivalent classes. Thus the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

can be diagonalized into

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A reflexive and transitive relation is a quasi-ordering. An anti-symmetric quasi-ordering is a partial ordering. A weak order is a quasi-ordering in which, for all x and y, either $x\mathrm{R}y$ or $y\mathrm{R}x$ (or both). A dominance relation is anti-reflexive and anti-symmetric. Dominance relations are particularly useful to sociologists and psychologists, and may even be applied to athletic teams in league play, chess tournaments, etc.

All of these concepts make certain demands on the corresponding matrices. Of all matrices of order n, how many represent an equivalence relation, how many a quasi-ordering, how many a dominance relation, etc.?

Chromatic Graphs

Let each of n points be connected to each of the other points, each line to be painted either red or blue. An equivalent system is the consideration of a group of n persons, any two of whom must be acquaintances or strangers.

The symmetric irreflexive matrix R can be formed, unity denoting joining by a red line, zero a blue line. A similar matrix B describing the blue joins can be formed. In this way the sum of R, B and the unit matrix consists entirely of unities. Some of the results are developed naturally by the matric representation. These include: the number of red lines is given by $\mathrm{tr}(R^2)/2$, where $\mathrm{tr}(R^2)$ is the sum of the diagonal elements of R^2, the number of red triangles is given by $\mathrm{tr}(R^3)/3!$, and the minimal number of uni-colored triangles can be found. Companion matrices such as R and B have some interesting properties, not yet fully exploited.

Again, unanswered questions arise. How many sets of four points are connected by lines of the same color? What if three or more colors are used? Can these matrices apply to map coloring?

Traversing a Network

Consider again a set of n points, for example the set of points at the centers of the squares of a chessboard, and a chess piece allowed to move in a prescribed manner, depending on its position on the board. An $n \times n$ matrix can be determined with unity showing the possible moves from each position. For example, we might have seven squares (Figure 2), numbered from one to seven. From an even numbered square we can move one square vertically or horizontally, from an odd numbered square one square diagonally.

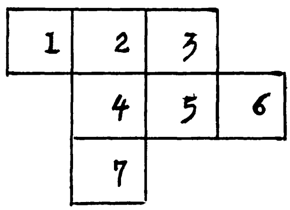

Figure 2.

The corresponding matrix is

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In how many ways can one move from 7 to 3 in four moves? In less than 6 moves? In how many ways can a piece return to its starting point in 5 moves? The answers are found by examining the element in the seventh row, third column of M^4, of the same element of $M + M^2 + M^3 + M^4 + M^5$, and $\mathrm{tr}(M^5)$, respectively.

Interesting and difficult questions immediately arise. In how many ways can a piece move, one square at a time vertically or horizontally, from one corner of an $p \times q$ chessboard without traversing any point more than once? Can winning combinations and positions in chess be predicted?

In conclusion, such matrices are not trivial. Further investigations can be carried out by amateur and professional alike. These investigations will result in more difficult questions, but such is the progress of mathematics.

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2. R. E. Greenwood and A. M. Gleason, "Combinatorial relations and chromatic graphs," Canadian J. Math., vol. 7, 1955.
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It took nearly three hundred years of mathematical thought to reach the modern abstract logical point of view. We do not begin our geometry by defining points, nor do we omit all definitions because it is perfectly clear what they are. We do not much care what they are, provided we may make certain purely logical assumptions about them. We are little interested in the truth of axioms, but much in their independence.

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A PROBLEM IN RELATIVITY

A. K. Rajagopal

The purpose of this note is to give an alternate proof to a theorem in relativity.

Theorem: If v_1, v_2, v_3 are the velocities of three frames of reference which move in the same direction, then

$$v_1 \cdot v_2 \cdot v_3 = \frac{v_1(v_2^2+v_3^2)+v_2(v_3^2+v_1^2)+v_3(v_1^2+v_2^2)+2v_1v_2v_3}{\sqrt{(1-\frac{v_1^2}{c^2})(1-\frac{v_2^2}{c^2})(1-\frac{v_3^2}{c^2})}}$$

where c is the velocity of light.

Proof: Though there is a direct but involved method to prove this, the following procedure is presented because of its neatness.

Let the transformation matrix be represented by

$$A(v) = (1-\frac{v^2}{c^2})^{-1/2}\begin{bmatrix} 1 & -v \\ -v/c^2 & 1 \end{bmatrix} \quad \text{(Lorentz Group).}$$

Now this is a continuous group in the sense that $A(v_1)A(v_2) = A(v_3)$ where the relations between the v's shall be derived.

We have

$$A(v_i) = (1-\frac{v_i^2}{c^2})^{-1/2}\begin{bmatrix} 1 & -v_i \\ -v_i/c^2 & 1 \end{bmatrix}, \quad i = 1, 2, 3 .$$

By the continuity of the group $A(v)$, on multiplying the matrices we arrive at

(a)
$$\frac{(1+\frac{v_iv_j}{c^2})}{\sqrt{(1-\frac{v_i^2}{c^2})(1-\frac{v_j^2}{c^2})}} = \frac{1}{\sqrt{(1-\frac{v_k^2}{c^2})}} \qquad \begin{array}{r} i = 1, 2, 3 \\ j = 2, 3, 1 \\ k = 3, 1, 2 \\ \text{in this order} \end{array}$$

and also

(b)
$$\frac{(v_i+v_j)}{\sqrt{(1-\frac{v_i^2}{c^2})(1-\frac{v_j^2}{c^2})}} = \frac{v_k}{\sqrt{(1-\frac{v_k^2}{c^2})}} \qquad \begin{array}{r} i = 1, 2, 3 \\ j = 2, 3, 1 \\ k = 3, 1, 2 \\ \text{in this order} \end{array}$$

Multiplying the three expressions found in (b) we get the result. The expression (a) is proved in Herbert Dingle's "Special Theory of Relativity" by a very difficult method. (This book is a Methuen Monograph printed in London). Also it may be stated that the continuity of the Lorentz Group $A(v)$ is given to be proved in the book – Walter Ledermann's Introduction to the Theory of Finite Groups (Oliver & Boyd, London) pp. 26-7.

BC 152 Camp, Belgaum, South India

FINITE SURFACES
A STUDY OF FINITE 2-COMPLEXES

E. F. Whittlesey

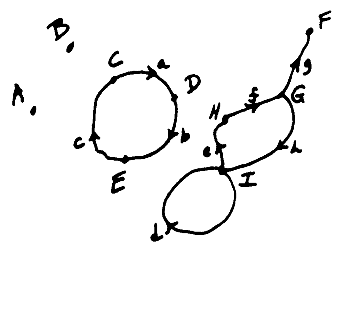
Part I. Local Structure.

The classification of closed or bounded 2-manifolds is common knowledge. It may be found in a paper by R. C. James [1] following lectures of A. W. Tucker. We extend the classification by finding a canonical form for an arbitrary finite 2-complex. We first review the analogous form for linear graphs. We define 2-complexes and characterize their local structure. In a second paper, we shall obtain the canonical form using the operations set forth here.

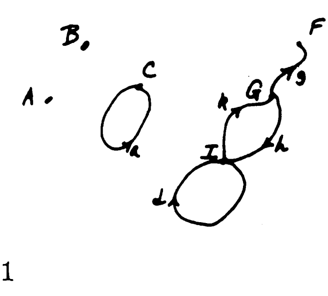
Given a finite set of points, called vertices, i.e. a 0-complex, a *linear graph (network* or *1-complex)* is obtained by assigning the ends of a finite number of segments to the vertices. We allow an edge in the resulting graph to have just one vertex, and then call it *cyclic* or a *loop*. One may operate on a network by subdividing its edges, or by the reverse process of *composition*. Two graphs are called (combinatorially) *equivalent* iff they differ only by a finite number of subdivisions and compositions. Of course, combinatorial and topological equivalence amount to the same (for graphs). One can reduce a graph to *canonical* form by composing edges until the process is no longer feasible. This removes all vertices of degree two, except on certain components which are loops, whereon one vertex is kept. Two graphs are *isomorphic* if there is a 1-1 incidence- and dimension-preserving correspondence of vertices and edges; if the two graphs are oriented, i.e. all edges are oriented, the isomorphism is required to preserve orientation also. Then it is evident that two graphs are equivalent iff they have isomorphic canonical forms. Notice that the canonical form is a minimal cellular partition; vertices of degree $\neq 2$ are never lost, but are of a singular nature.

Example.

Graph.



Canonical form.



Vertices.		Edges.		Vertices.		Edges.	
A	B	$Da^{-1}C$	$Cc^{-1}E$	A	B	CaC	$Gk^{-1}I$
C	D	DbE	GgF	C	F	GgF	IdI
E	F	HfG	$He^{-1}I$	G	I	GhI	
H	G	GhI	IdI				
I							

Turning to two dimensions, we speak, instead of networks, of *surfaces,* more precisely, of *surfaces with or without singularities,* or *finite 2-complexes,* or *finite 2-dimensional cell complexes.*

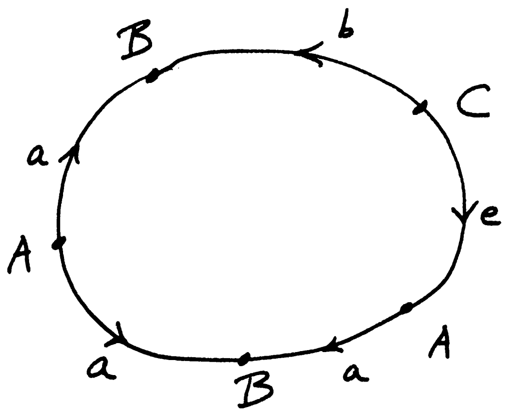
Let an oriented 1-complex be given, the *1-skeleton* of the 2-complex, K, to be constructed. Let a finite set of disjoint plane disks (closed 2-cells) be given. To the periphery of each disk is assigned some closed edge path in the 1-skeleton, and that path may or may not be a single vertex. If we now identify periphery of disk and assigned path we get a *2-complex, K.*

The interior of a disk is an open 2-cell. Thus K partitions into 0-, 1-, and 2-cells, and we speak of a *cellular* partition (of the space) of K. A 2-complex is *oriented* if all the cells are oriented. Two special kinds of cell complexes are of particular importance: a *polygonal* complex is a complex wherein the boundary of each 2-cell is a simple closed path (not a vertex), the 1-skeleton is simplicial, and any two 2-cells have at most one face in common; a *simplicial* 2-complex is a complex which is polygonal and every 2-cell is a triangle. We shall refer to the 2-cells and assigned boundary paths as *singular* polygons.

In order to specify a 2-complex, we need, in addition to the list of vertices and list of edges for the 1-skeleton, a list of the boundary paths of all the 2-cells. These are closed edge-paths, $AaBb^{-1}C\cdots MmA$, possibly degenerate, A; some of the paths may be repeated several times, perhaps along with some of the inverse paths also – each occurrence corresponding to the boundary of a different 2-cell. Instead of paths, we call these *words,* for they correspond 1-1 to the 2-cells and define a mapping of the original disks.

For the present, we shall suppose that every cell is face of some 2-cell; then we can omit the lists of vertices and edges, since they can be read off from the system of words.

Our definition of 2-complex presupposes an oriented 1-skeleton; but an edge, b, can be reoriented, and this involves replacing b by b^{-1} and vice versa, throughout the system of words. Also, we can change variables, i.e. rename a vertex or edge with a new letter, and this involves the same change throughout the system of words. If a singular polygon is given, $AaBb^{-1}CeAaBa^{-1}A$, it is clearly immaterial which vertex is chosen as initial point; thus we may allow cyclic permutation of the edges, e.g. in



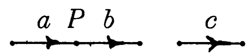
the above to $CeAaBa^{-1}AaBb^{-1}C$. Furthermore, orientation of a polygon is immaterial, or, to put it another way, it makes no difference whether we travel around the disk in a clockwise or counterclockwise fashion. Hence we can replace any word by its inverse; for example, the above last becomes $CbBa^{-1}AaBa^{-1}Ae^{-1}C$. These changes in representation we regard as formal equivalences.

Two systems of words for a complex, K, are *combinatorially* equivalent, or, simply, equivalent, iff a finite sequence of (formal equivalences and) subdivisions and compositions of edges and polygons will carry one system into the other.

This involves the following changes in the system of words.

(1). Subdivision and composition of an edge:
the substitution

$$\begin{cases} aPb \longleftrightarrow c \\ b^{-1}Pa^{-1} \leftrightarrow c^{-1} \end{cases}$$

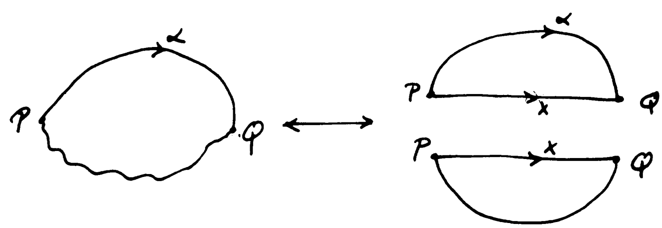


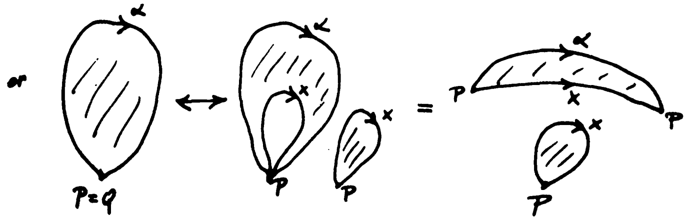
is admissible, provided, in going from left to right, a, P, b occur only in the sequence aPb or $b^{-1}Pa^{-1}$ and then all sequences of this form must be replaced if any one is, and c must be a new letter not in the system; in going from right to left, a, P, b must be new symbols not in the system.

(2).. Subdivision and composition of polygons.
the substitution (for a word)

$$\cdots P\alpha Q\cdots \leftrightarrow \begin{Bmatrix} \cdots PxQ\cdots \\ Qx^{-1}P\alpha Q \end{Bmatrix}$$

is admissible, provided that in going from left to right x is a new





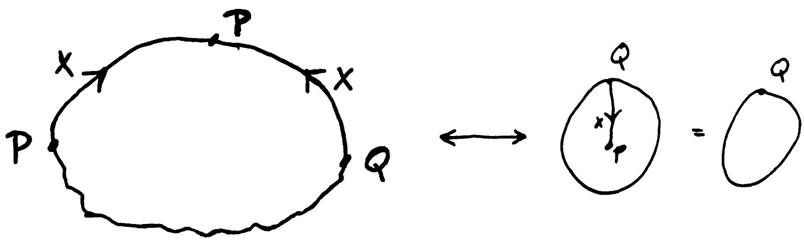
symbol and that in going from right to left, x and x^{-1} occur nowhere else in the system. (The dots may represent empty sets, and α may be empty, and it may be that $P = Q$.)

Consequence. *Incisions*.

The following operation on a word is admissible:

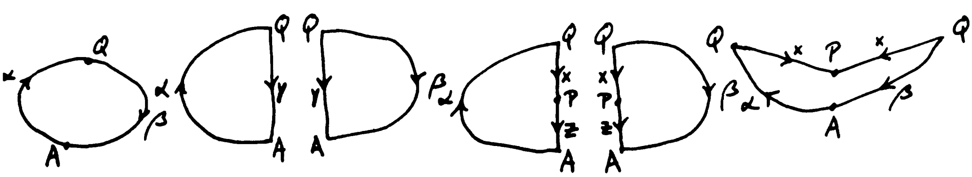
$$\cdots QxPx^{-1}Q\cdots \longleftrightarrow \cdots Q\cdots$$

provided x, P, x^{-1} occur nowhere else in the system of words.



Proof.

$$A\alpha Q\beta A \longleftrightarrow \begin{array}{c} A\alpha QyA \\ Ay^{-1}Q\beta A \end{array} \longleftrightarrow \begin{array}{c} A\alpha QxPzA \\ Az^{-1}Px^{-1}Q\beta A \end{array} \longleftrightarrow A\alpha QxPx^{-1}Q\beta A$$



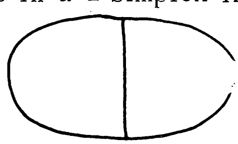
To determine the global structure of a complex we need to have first a knowledge of the local structure, of behaviour in a neighborhood of a point; with this understanding of local structure, we can solve the problem of recognition of a 2-complex by putting the pieces together, as it were. In the case of graphs, the problem of local structure is completely resolved by a knowledge of the degree of a vertex. The matter is less simple for surfaces.

To simplify our discussion of neighborhoods, we suppose a simplicial 2-complex given – if a given 2-complex is not simplicial, it can be made so with two, or in the extreme case, with three barycentric subdivisions. (The three subdivisions will be needed if some polygon has a boundary path consisting of a single point.) Note that continued barycentric derivation preserves the simpliciality of a partition, and that the diameter of a simplex can be made as small as we like by refining the partition often enough.

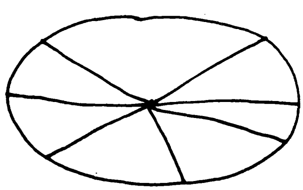
The *neighborhood* of a point is the sum of the cells which have the cell, containing the point, as face. By adding the faces of these cells we can close the neighborhood.

The Neighborhoods

(1). The simplest and most typical neighborhood would seem to be a 2-cell neighborhood. A point with a neighborhood homeomorphic to a 2-cell is *regular*. In addition to the points in the 2-simplexes, the set of regular points includes the points in a 1-simplex if the 1-simplex is face of exactly two 2-cells.

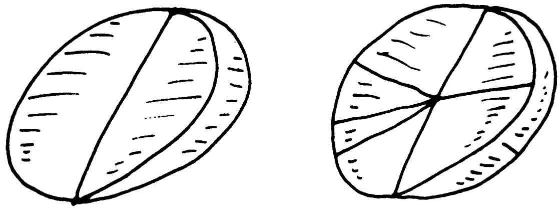


It may occur that some vertex is regular. Then its neighborhood looks like the picture below: a disk with the vertex at the center and several radii emanating to the periphery.



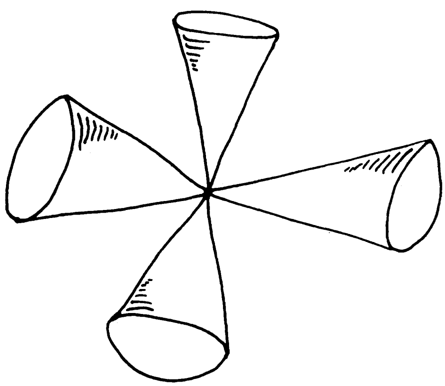
The points of a complex which are not regular are *singular*. All singular points must lie in the 1-skeleton; however, as we have just seen, not all the points in the 1-skeleton are necessarily singular.

(2). If we identify the x-axes of n copies of the euclidean ½-plane $y \geqq 0$, we obtain a space called a *book*; the open ½-planes $y > 0$ become the *leaves* of the book; the x-axes $y = 0$ unite to form the *line* of the book. A point with a book neighborhood having $n \neq 2$ leaves is a *line* singularity. In any case, we call n the *degree* of the line. If the line has degree 2, it is regular; if it has degree 0, the line is not face of any 2-cell.



The line of a book is a topological 1-cell, of course. In the figure above we have two books, each of three leaves. One is the star of a 1-simplex, the other is the star of a vertex. In the second case, the leaves of the book may consist of more than one 2-simplex each.

(3). If we identify the origins of n copies, $n > 0$, of the euclidean plane, we obtain a space called a *cone*; the origins combine to form the *center* of the cone, while the part of a plane different from the origin becomes a *leaf* of the cone. A point in a 2-complex with a conical neighborhood having more than one leaf, is a *conical* or *isolated singular* point. A conical point is necessarily a vertex. The motivation of the second term is, of course, that the center is the only singular point in the neighborhood.



(4). A singular point which is not isolated or line-singular is called a *node*. A node is necessarily a vertex.

Before describing the neighborhood of a node, we interpose some needed definitions. The singular points of a 2-complex occupy a 1-subcomplex called the *singular graph*. The complement of the singular graph consists of the regular points; this complement falls into components called *surface components*. The nodes and isolated singular points are called, collectively, the *point-components* of a complex K. If S denotes the singular graph of K, and if P is the set of point components of K, then $S-P$ is a subspace of K whose components we call *line components* of K. The line components of K are, topologically, circles or open 1-cells.

The neighborhood of a point in K is a subcomplex of K which partitions into surface, line, and point components which we call *local.* Thus the neighborhood of a regular point consists of a single local surface component – a topological 2-cell; the neighborhood of a conical point consists of a single local point component – the conical point itself – and several local surface components – the leaves of the cone – each homeomorphic to a punctured plane; the neighborhood of a line singularity consists of one local line and several ($n \neq 2$) local surface components – the leaves of the book, each a topological 2-cell.

We can now describe the neighborhood of a node, N. The only local point component at a node, N, is N itself, since the other points of the neighborhood of N lie in 1-cells or 2-cells incident with N, and N is not regular or line-singular. The local lines at N are certain 1-simplexes consisting of line singularities of the total complex. These local lines emanate from N, and such local lines exist, since N is a node, not an isolated singularity. Let us consider the local surface components at N, and reconstruct them from the 2-simplexes incident with N. A 2-simplex incident with N has two edges incident with N. They may be regular or line-singular. If one of these edges is regular, compose the 2-simplex with the one other uniquely determined 2-simplex having that regular edge. Continue the composition process until the local surface component is exhausted. It may be that this local surface component turns out to be the leaf of a cone of center N. N is a node, however, and there must be a local surface component at N which contains a local line incident with N. If we start the reconstruction of a local surface component from such an edge, there are just two possibilities: after a finite number of compositions we either arrive back at the same local line or at a different local line. If we arrive back at the local line where we started, we call the local surface component obtained a *cornet.* If we arrive at a different local line, we call the local surface component a *fan* at N. Thus the neighborhood of a node consists of a cone of center N and a set of fans and cornets of center N. The neighborhood of N is entirely prescribed by specifying:

(a) the number $n \geqq 0$ of cone leaves;

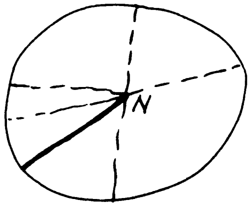
(b) the number of fans;

(c) the number of cornets;

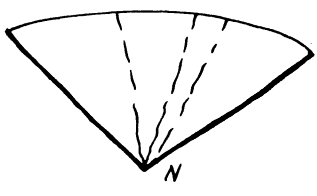
(d) the number of local lines;

(e) the correspondence of the local lines to the fans and cornets.

There are clearly restrictions: if there is a cone at N, there must also be



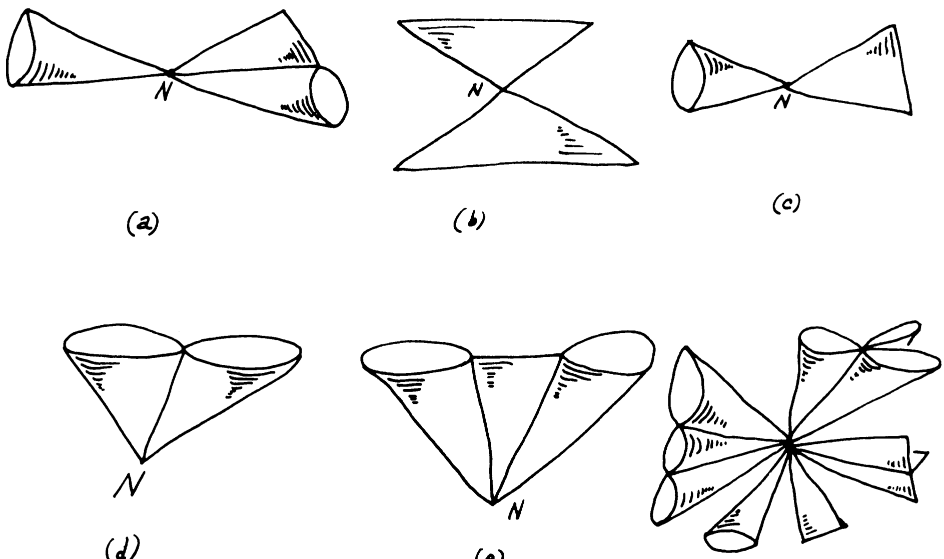
Cornet



Fan

fans or cornets; if there is a cornet at N, there must be a fan or cornet with the same local line; if there is a fan, then there must also be other surface components at N (otherwise N were a line singularity).

Examples of neighborhoods of nodes.



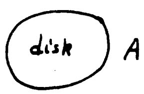
(a) (b) (c) (d) (e) (f)

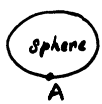
Of particular interest is a *point of articulation*: this is a point whose neighborhood is disconnected by its removal. Of course, a conical point is articulate, otherwise an articulation is a node. (a), (b), (c), (f) above are examples of articulations. A point is *in*articulate if it is not articulate.

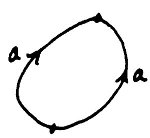
The property of a vertex of being inarticulate or articulate has an algebraic as well as a geometric significance. In the system of words for a complex, wherever an edge symbol, x, occurs, it is preceded by a certain vertex A, and x^{-1} is always followed by A; vice versa, for the vertex symbol which follows x. Thus the various occurrences of the vertex symbol A fall into equivalence classes as follows: two occurrences of A are equivalent if there is a finite sequence of occurrences of A joining them such that for each two successive occurrences of A there is an edge symbol, x, such that A precedes x in two occurrences of x, or A follows x in two places, or one occurrence of A precedes x and the other follows x, or vice versa. If all occurrences of A lie in one class, A is inarticulate, otherwise A is articulate. This means that the identification of several vertices to form an inarticulate vertex is specified completely by the way in which the edges of a polygon are attached to the 1-skeleton; that is, when a complex is constructed by mapping boundaries of polygons into a 1-complex, the image of a vertex of a polygon, sent onto an inarticulate vertex, is known entirely by the fact that that image is the end of certain edges—it is obtainable from the incident edges by continuity of the mapping. But in the inarticulate case, not so. Thus, a complex with points of articulation can be obtained from a complex without points of articulation by identifying certain vertices, but not the edges incident therewith.

The one exception to the rule, is the sphere. It can be represented by a single capital letter. But otherwise, our notation can be vastly simplified by the omission of the capitals for the inarticulate vertices, their position being specified entirely by the above relation of "equivalent position" in the system of words. We use this simplification at every opportunity. We must keep, then, except in the case of the sphere, just the capitals for the articulate vertices.

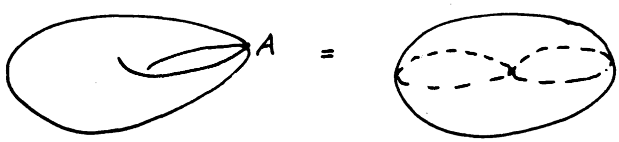
Examples.







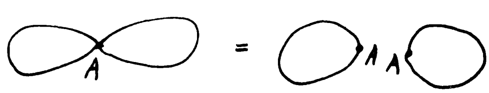
A sphere can be represented by a single capital : A. This indicates that a sphere can be obtained by identifying all the points on the periphery of a disk. A sphere can also be obtained from a digon : aa^{-1}. However, the first representation is a minimal cellular partition – 0-cell and 2-cell with no intervening edge. A pinched sphere is obtained by identifying two points on the sphere. The resulting single point is a point of articulation – a conical point. The pinching process can be easily indicated by the symbolism : the sphere, $AaBa^{-1}A$, becomes the



pinched sphere by setting $A = B : AaAa^{-1}A$. Similarly, we can represent two tangent spheres by simply writing down two copies of the capital A.

A

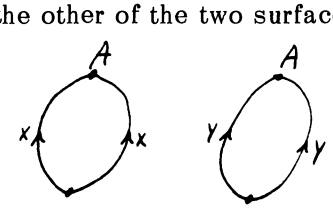
A



Each capital represents a surface component. We could also, as above, make an incision on one or the other of the two surface components :

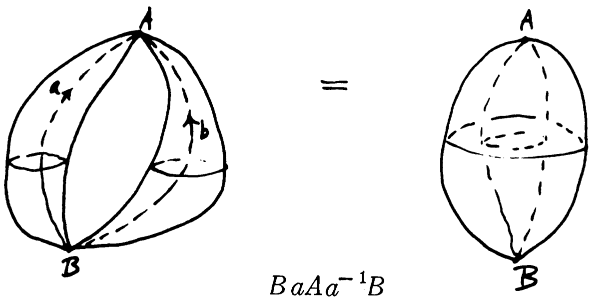
xAx^{-1}

yAy^{-1}



Again, A is a conical point.

Two spheres with two common points are represented below.



$BaAa^{-1}B$

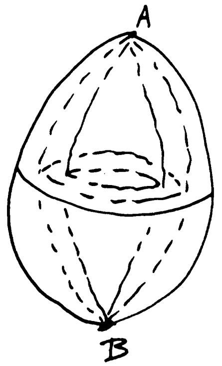
$BbAb^{-1}B$

A and B are both conical. In a similar fashion, one can describe an "onion" of many layers:

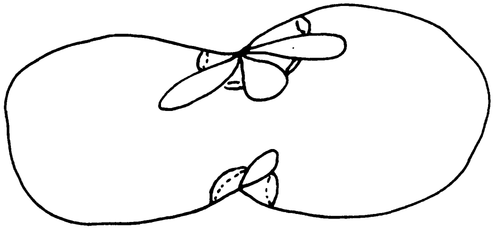
$Ba_1Aa_1^{-1}B$

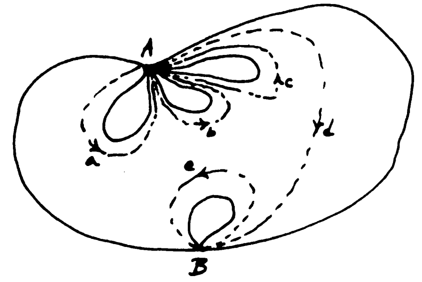
$\cdots$

$Ba_nAa_n^{-1}B$



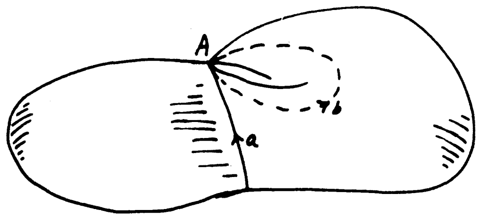
A sphere with several pinches in different points is illustrated below. The picture on the right shows cuts made to make a representation by a singular polygon.





A and B are both conical points. $AaAbAcAdBeBe^{-1}Bd^{-1}Ac^{-1}Ab^{-1}Aa^{-1}A$.

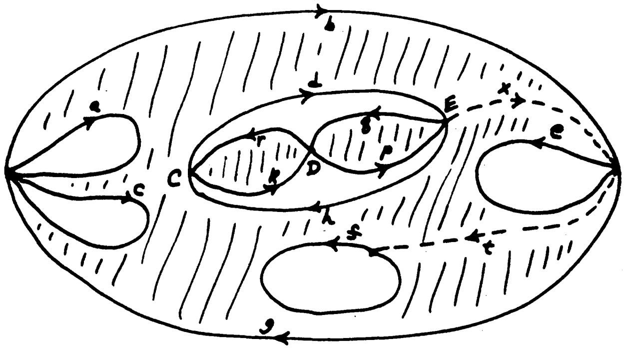
Specify a segment on each of two spheres and identify the segments; identify another point on one of the spheres with an end of this segment:



$$aAa^{-1} \qquad aAbAb^{-1}Aa^{-1}$$

A is a node and is articulate. a is a line; all points on a are line singularities. The singular graph is BaA, where B denotes the initial point of a. B is a node but is inarticulate, so we have left it out of the words and the diagram.

Below is illustrated a plane complex.



With the indicated partition, each surface component is represented by a single singular polygon, and thus the system of words consists of just three words:

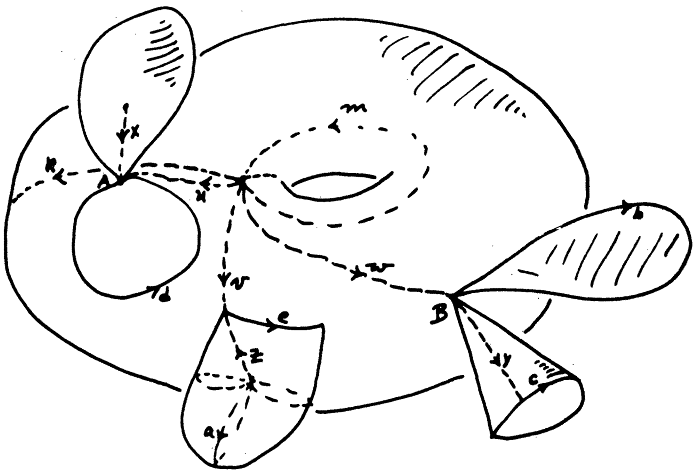
$$Cr^{-1}Dk^{-1}C$$

$$DpEqD$$

$$AbBx^{-1}Ed^{-1}Ch^{-1}ExBeBtft^{-1}BgAc^{-1}Aa^{-1}A$$

A, B, C, D, E are nodes and are articulate. All edges represent lines, except x and t.

As a final example we consider a bounded torus. To a point A of its boundary is attached a sphere. To a regular point B of the torus is attached a disk at a boundary point of the disk, and also a second disk by a regular point of the disk. To a sigment e disjoint from the boundary d and the point B is attached a projective plane.



$$xAx^{-1}$$
$$Bycy^{-1}B$$
$$BbB$$
$$aazee^{-1}z^{-1}$$
$$mkm^{-1}k^{-1}uAdAu^{-1}vee^{-1}v^{-1}wBw^{-1}\ .$$

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[2] E. F. Whittlesey, *Classification of finite 2-complexes*, Proc. AMS, vol. 9, no. 6 (1958) pp. 841-845.

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MATHEMATICAL HEREDITY

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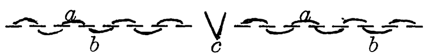
In order to explain the principle of Mathematical Heredity it will be necessary to introduce the Magic Four-Square, since the principle is particularly applicable to Magic Squares. In general, for any Four-Square, the sixteen cells must be filled with numbers (or algebraic quantities) in such a manner that the sums in all four rows, four columns, and two principal diagonals, shall be equal. Then, from these ten defining equations, it may be proved that the six secondary diagonals have this same sum.

In order to form a four-square the Magic Series must be of the form shown below. (Series A). This is the least restricted series possible.

Series (A). 

Here all the eight a-differences must be alike, all the six b-differences must be equal, while the c-difference in the middle may be different from the other two. Such a series is the following one.

Series (B). 5 8 7 10 9 12 11 14 16 19 18 21 20 23 22 25

This series will form a four-square in 432 different ways, and the squares will be found to fall into six distinct classes in which the properties regarding symmetry are somewhat different. One of these classes is particularly interesting, and it has been named the "Diabolic" Four-Square. All Magic Four-Squares may be derived from one "Primitive" four-square by means of operators, or by following a Magic Figure in distributing the terms of a Magic Series such as (A) or (B) in proper order in the sixteen cells of the four-square.

There are forty-eight Magic Figures of the Diabolic type, and one of these is of singular importance. Figure 1 represents "Diabolo" a Devil with pointed chin and pointed ears that stand erect as he sits most uncomfortably on his pyramidal throne. With legs outstretched and wings spread he moves his arms frantically up and down.

The order in which the numbers of the Magic Series are set down by means of the Magic Figure is as follows. Numbering the cells of the four-square in order from the upper-left, place the first term of the series in cell 9 at Point P. The successive terms of the series are then set in cells 4, 7, 14, 15, 6, 1, 12, 2, 11, 16, 5, 8, 13, 10, and 3.

General Properties of the Diabolic Four-Square.

1) The four rows, four columns, and eight diagonals must each have the same sum S.

2) The corners of the four-square have the same sum S, likewise the corners of all four three-squares; also, the four numbers in each of the nine two-squares, and the four end-numbers of the six two-by-four rectangles.

In the case of the Series (B) distributed by the Magic Figure 1, the

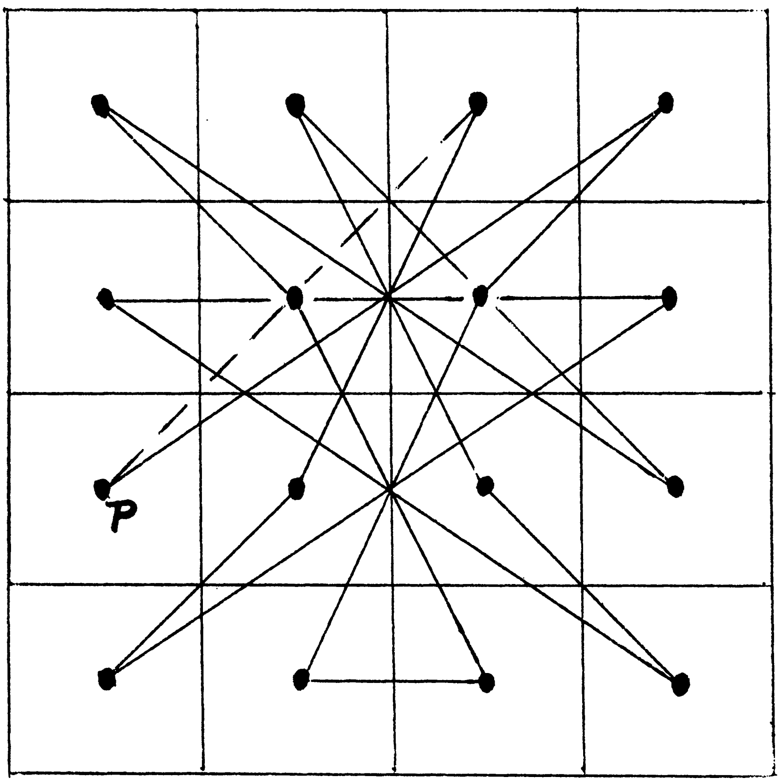


Figure 1.

sum will be found to be 60 for each of the specifications above.

Now what has been said about the capabilities of Diabolo is only an introduction. For a further discussion of these properties let us adopt the first sixteen integers, for an arithmetic progression is merely a restricted case of the Magic Series (A). When these are placed in the sixteen cells of the four-square in the order determined by the Magic Figure we get the Diabolic Square. (Fig. 2). Inspection will show at once that all of the properties of the Diabolic Four-Square are fulfilled in Figure 2.

Next construct in a similar manner fifteen more similar Diabolic squares using the consecutive integers from 17 to 32, 33 to 48, 49 to 64, etc., and finally 241 to 256. Lastly, arrange these sixteen first-order Diabolic squares into a second-order square sixteen times as large as the first ones, distributing in the order indicated by the Magic Figure. This produces Figure 3 in which each first-order square is itself Diabolic, and furthermore the six-

7	9	16	2
12	6	3	13
1	15	10	8
14	4	5	11

Figure 2.

teen rows, sixteen columns, and thirty-two diagonals of the large square

103	105	112	98	135	137	144	130	247	249	256	242	23	25	32	18
108	102	99	109	140	134	131	141	252	246	243	253	28	22	19	29
97	111	106	104	129	143	138	136	241	255	250	248	17	31	26	24
110	100	101	107	142	132	133	139	254	244	245	251	30	20	21	27
183	185	192	178	87	89	96	82	39	41	48	34	199	201	208	194
188	182	179	189	92	86	83	93	44	38	35	45	204	198	195	205
177	191	186	184	81	95	90	88	33	47	42	40	193	207	202	200
190	180	181	187	94	84	85	91	46	36	37	43	206	196	197	203
7	9	16	2	231	233	240	226	151	153	160	146	119	121	128	114
12	6	3	13	236	230	227	237	156	150	147	157	124	118	115	125
1	15	10	8	225	239	234	232	145	159	154	152	113	127	122	120
14	4	5	11	238	228	229	235	158	148	149	155	126	116	117	123
215	217	224	210	55	57	64	50	71	73	80	66	167	169	176	162
220	214	211	221	60	54	51	61	76	70	67	77	172	166	163	173
209	223	218	216	49	63	58	56	65	79	74	72	161	175	170	168
222	212	213	219	62	52	53	59	78	68	69	75	174	164	165	171

Figure 3.

have the same sum, 2056. Figure 5 shows the sums in the first-order squares by rows, columns, and diagonals.

But besides all these properties the Conditions 2 for the Diabolic are obeyed. Taking the sixteen sums of all the numbers in the first-order four-squares we produce Figure 4 in which the various properties of the Diabolic First-order squares are followed precisely, even to the sum of the corners of the large square, the sum of the four corners of each three-square, and

1672	2184	3976	392
2952	1416	648	3208
136	3720	2440	1928
3464	904	1160	2696

Figure 4.

the four corners of each two-square, as well as the ends of each two-by-four rectangle. The sum in all cases is 8224.

418	546	994	98
738	354	162	802
34	930	610	482
866	226	290	674

Figure 5

It might be supposed that this giant offspring of Diabolo is his largest and most remarkable, but such is not the case. Let us look at his third-order progeny. Imagine sixteen successive squares similar to Figure 3. Now the first one will contain the natural numbers from 1 to 256, the second those from 257 to 512, etc. The last one will run from 3841 to 4096. Let these be set into a third-order square, their positions being determined as before by the Magic Figure, Diabolo. We shall then have a square with sixty-four numbers on each side, in all 4096 integers.

Each of the 256 first-order squares which make up this third-order monster are totally Diabolic, since they are constructed precisely like Figure 2. The sixteen second-order squares are constructed precisely in the same form as Figure 3, and so they are completely Diabolic. The Conditions No. 2 for Diabolic squares are satisfied by considering each first-order square a unit cell in the second-order square. To explain the properties of the entire third-order square it is necessary to construct Figure 6.

In this case each unit square is similar to Figure 3, a second-order square. The upper number shown in each square is the sum of each row,

26632 106528	34824 139296	63496 253984	6152 24608
47112 188448	22536 90144	10248 40992	51208 204832
2056 8224	59400 237600	38920 155680	30728 122912
55304 221216	14344 57376	18440 73760	43016 172064

Figure 6.

column, and diagonal for that particular second-order square. The lower number is the total sum of all the integers in the particular second-order square. Thus, it may be seen by adding the upper numbers that all rows, columns, and diagonals of the third-order square give a sum of 131,104. Also, taking each of sixteen squares of Figure 6 as a unit cell, we find that the sum of all rows, columns, and diagonals of these squares is 524,416 in each case. Thus, the sum of the corners of the four-square may be found, the sum of the corners of each three-square, the sum of the corners in each such two-square, and the ends of each two-by-four rectangle. In all cases the sum is 524,416. Hence, the third-order square is Diabolic in all its parts, first-order, second-, and third-order.

So, as the order of the square increases, all the properties of the constituent lower orders are retained, and in addition by taking the next lower order of square as the unit cell it will follow that all the properties of the largest square are strictly Diabolic. Thus it is that Diabolo generates a progeny of Super-giants, giants that grow larger and more complex without limit, and in all cases Diabolo transmits his characteristics unto the third and fourth generations, and toward infinity; and this by a property of the Magic Figure that will be called Mathematical Heredity.

But this is not all that may be said of the amazing Diabolo, for our Magic Series (A) may be replaced by a Multiplicative Series in which we replace the three a, b, and c-differences by three *ratios*. All of the properties of the Diabolic squares just shown then become true for a new set of squares in which ratios are substituted for differences. Hence, a geometric progression distributed by the use of the Magic Figure will form a whole progeny of Multiplicative Squares with properties analogous to those formed by the Arithmetic Series, and these too by the process of

Mathematical Heredity.

Indeed, it may be said that this property is characteristic of all magic squares. All that is needed is a Magic Series and a corresponding Magic Figure. From these will spring a progeny of Super-giants bearing the characteristics of the original first-order square, and these properties will be transmitted by Mathematical Heredity.

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167. The mathematician is an inventor, not a discoverer.

From *Remarks on the Foundations of Mathematics* by Ludwig Wittgenstein, The Macmillan Company, New York, 1956. Used with the permission of the publisher.

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to *Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.*

ARITHMETIC PROGRESSIONS OF N RELATIVELY PRIME INTEGERS

Sam Matlin

Euclid proved that there are infinitely many primes by exhibiting the number $1+2\cdot 3\cdot 5\cdot \cdots \cdot p_n$, which is either another prime itself, or contains another prime greater than p_n as a divisor. Implicit in his ingenious construction is yet another interesting idea. It can be shown that there are infinitely many arithmetic progressions which contain infinitely many primes (Dirichlet's Theorem states that if a is positive and if a and b are relatively prime, then there are infinitely many primes of the form $an+b$). A next logical conjecture would be: for any integer n, can n primes be found such that they form an arithmetic progression? Since this appears to be difficult to answer, one may weaken the conjecture by asking: can n integers which are relatively prime in pairs be found such that they form an arithmetic progression, for any integer n? This has been answered in the affirmative by Davis. [1] We shall prove in this note a slightly stronger form of this latter conjecture.

Theorem: For any integer n an arithmetic progression exists in which: a) the first two terms are primes; b) the first n terms are relatively prime in pairs; c) there are infinitely many primes in the progression.

Proof: For any given n, let π_n denote the product of the first n primes; let p_{n+1} denote the $(n+1)$st prime, and let m denote any integer which is relatively prime to p_{n+1} such that $p_{n+1}+m\pi_n$ is a prime (such an m will exist by Dirichlet's Theorem).

Consider the arithmetic progression:

$$p_{n+1},\ p_{n+1}+m\pi_n,\ p_{n+1}+2m\pi_n,\ p_{n+1}+3m\pi_n,\ \ldots,$$
$$p_{n+1}+im\pi_n,\ \ldots,\ p_{n+1}+(n-1)m\pi_n .$$

a) The first two terms are primes.

b) These terms are relatively prime in pairs, for if any two of these terms, say $p_{n+1}+im\pi_n$ and $p_{n+1}+jm\pi_n$, had a common divisor $d > 1$; they would have a common prime divisor, $q > 1$. If $q \mid (p_{n+1}+im\pi_n)$ and $q \mid (p_{n+1}+jm\pi_n)$, then q divides their difference: $q \mid (i-j)m\pi_n$. Therefore either $q \mid (i-j)$ or $q \mid m$ or $q \mid \pi_n$. But if $q \mid \pi_n$ and $q \mid (p_{n+1}+im\pi_n)$, then $q \mid p_{n+1}$. Therefore $q = p_{n+1}$, so that $p_{n+1} \mid \pi_n$. Since this is impossible, $q \nmid \pi_n$, and we may conclude that $q > p_n$. But since $(i-j) < n < p_n < q$, $q \nmid (i-j)$. Therefore $q \mid m$. But if $q \mid m$ and $q \mid (p_{n+1}+im\pi_n)$, then $q \mid p_{n+1}$, which is contrary to the hypothesis that $(p_{n+1}, m) = 1$. Therefore no such q can exist.

c) There are infinitely many primes in this progression, by Dirichlet's Theorem.

Corollary 1: Starting with 1 as first term we may find n integers which are relatively prime in pairs and which are in arithmetic progression.

Proof: Consider the arithmetic progression:

$$1\,,\quad 1+\pi_n\,,\quad 1+2\pi_n\,,\quad \cdots\,,\quad 1+(n-1)\pi_n\,.$$

By an argument similar to the previous one it may be shown that these n integers are relatively prime in pairs.

Corollary 2: Starting with any prime p as first term, we may find p relatively prime in pairs integers which are in arithmetic progression.

Proof: Consider the arithmetic progression:

$$p\,,\quad p+(\pi_p/p)\,,\quad p+(2\pi_p/p)\,,\quad \cdots\,,\quad p+((p-1)\pi_p/p)\,.$$

Again by an argument similar to the previous one it may be shown that these p numbers are relatively prime in pairs.

Remark: The procedure employed in the theorem will actually produce more than n such integers; in fact we may continue the progression up to $p_{n+1}+(p_{n+1}-1)m\pi_n$, so that we have p_{n+1} rather than only n such integers.

I am indebted to S. Stein for his proposal of the conjecture solved in this paper.

REFERENCE

[1] Martin Davis, *Computability and Unsolvability*, McGraw-Hill, New York, 1958, p. 45.

Brooklyn 25
New York

MATHEMATICS AND PHILATELY

Maxey Brooke

Stamps have been issued to honor presidents and kings, warriors and statesmen, poets and musicians, saints and villains; even chessplayers and stamp collectors have been commemorated.

In spite of their contributions to civilization and culture, only a handful of mathematicians have appeared on postage stamps; none on those issued by the United States or Great Britain. Perhaps this is a measure of the esteem held by the general public for things mathematical.

A few famous names are represented, but many are missing. Newton, Gauss, and Pascal, Laplace, Lie, and Poincare, to mention a scant half-dozen, are without philatelic honor.

Germany appears to have been the first country to put a mathematician's picture on a postage stamp. In 1926, she issued a 40 pfennig violet (360) with a picture of Leibnitz. There seemed to be no particular reason for this issue which had other German greats including Goethe, Schiller, Beethoven, and Bach. Twenty-four years later, a 24 pfennig red (10N66) featuring Leibnitz, as well as a 1 pfennig grey (10N58) featuring Euler were issued to commemorate the 250th anniversary of the founding of the Academy of Science in Berlin.

In 1928, Cyprus, celebrating its fiftieth year as a British colony, issued a set of stamps which included a 1 pi blue and black (115) with a picture of Zeno. This makes him the only ancient Greek mathematician so honored.

To Norway goes the credit for having issued the first set of stamps especially for a mathematician. In 1929, a 10 öre green (145), 15 öre brown (146), 20 öre red (147), and 30 öre ultramarine (148) set commemorated the centenary of Abel's death.

Chronologically, the next to issue a mathematical stamp was Hungary. A set of famous Hungarians included a 70 filler cerise (479) with a picture of Bolyai. Oddly enough it was Farkas Bolyai, the mathematics teacher rather than his more famous son Janos Bolyai who received recognition.

France has probably produced more great mathematicians than any other country. But from this embarrassment of riches, only one has appeared on a stamp. In 1937, two slightly different 90 cent reds (330, 331) featuring Descartes were issued in commemoration of the third centenary of the publication of "Discours de la Methode". Thus it was Descartes, the philosopher rather than Descartes the mathematician who was honored.

On the other hand, there is little question who was Ireland's greatest mathematician. In 1943 the Irish Free State issued a half-penny green (126) and a 2½ penny brown (127) depicting Hamilton's portrait. This was not issued in honor of the man but in honor of the 100th anniversary of his discovery of quaternions.

In 1946, Russia issued a 30 kopeck brown (1050) and a 60 kopeck grey brown (1051) in honor of the 125th anniversary of the birth of Chebichev. And in 1951, as part of a scientist series, a 40 kopeck brown (1575) featured Lobochevski.

Nine mathematicians in all. A poor representation of the thousands who have labored in the field. Of course, there are men who contributed to mathematics but who were not primarily mathematicians; Düer, Copernicus, Roemer, Tycho Brahe, Galileo, and Chapylgin. These men have their stamps. And da Vinci has had no fewer than 17 stamps issued by six countries.

I don't know what this proves. I submit it only as information that might prove interesting. The numbers in parentheses are the Scott Stamp Catalogue numbers.

Sweeny, Texas

A Modern Approach

To add one and one
Used to be fun.
To-day I'm unsure
What sum to secure
I'm told it might even be none.

BARNET RICH

in Jan. 1960 issue of Brooklyn Technical High School Math Student.

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Joseph Seidlin, Alfred University, Alfred, New York.*

ROW RANK AND COLUMN RANK OF A MATRIX

Stephen A. Andrea and Edward T. Wong

(a_{ij}) is an $m \times n$ matrix over a field T. V and U are vector spaces over F with $e_1, e_2, \cdots, e_n$ and $f_1, f_2, \cdots, f_m$ as their bases respectively.

Consider

$$X = \{x_i\} \quad \text{where} \quad x_i = \sum_{j=1}^{n} a_{ij} e_j \,, \quad i = 1, 2, \cdots, m$$

$$Y = \{y_j\} \quad \text{where} \quad y_j = \sum_{i=1}^{m} a_{ij} f_i \,, \quad j = 1, 2, \cdots, n \,.$$

X is a subset of V and Y is a subset of U. The row rank of (a_{ij}) is defined to be the dimension of the subspace spanned by X and the column rank of (a_{ij}) is defined to be the dimension of the subspace spanned by Y. It is very easy to verify that the row rank and the column rank of the matrix are independent of the choice of U and V and their bases.

Let $\{\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_p\}$ be a maximal linearly independent set of X, and $\{\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_q\}$ be a maximal linearly independent set of Y. Hence the row rank of (a_{ij}) is p while the column rank of (a_{ij}) is q. With no loss of generality, we can assume $\overline{y}_j = y_j$, $j = 1, 2, \cdots, q$.

Let

$$y_j = \sum_{k=1}^{q} \alpha_{kj} y_k \,, \quad j = 1, 2, \cdots, n \,, \quad \alpha_{kj} \in F \,.$$

$$y_j = \sum_{i=1}^{m} a_{ij} f_i = \sum_{k=1}^{q} \propto_{kj} y_k = \sum_{k=1}^{q} \propto_{kj} \left(\sum_{i=1}^{m} a_{ik} f_i \right) = \sum_{i=1}^{m} \left(\sum_{k=1}^{q} \propto_{kj} a_{ik} \right) f_i$$

Therefore $a_{ij} = \sum_{k=1}^{q} \propto_{kj} a_{ik}$ for all $i = 1, 2, \cdots, m$ and all $j = 1, 2, \cdots, n$. Any x_t in $\{x_i\}$,

$$x_t = \sum_{j=1}^{n} a_{tj} e_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{q} \propto_{kj} a_{tk} \right) e_j = \sum_{k=1}^{q} a_{tk} \left(\sum_{j=1}^{n} \propto_{kj} e_j \right).$$

x_t is a linear combination of q vectors. Therefore $p \leqq q$.

By these same methods, we can show that $q \leqq p$. Hence we have proved the row rank of (a_{ij}) equals the column rank of (a_{ij}).

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MORE ABOUT THE NORMAL EQUATION OF THE LINE $Ax+By+C=0$

C. N. Mills

I. Many methods are known for the derivation of the normal equation of the line $Ax+By+C=0$, however the following method may not be widely known. Let θ be the inclination of the given line to the X-axis, and ϕ the inclination of the normal to the X-axis. Rotating the axes through the angle ϕ, we immediately find the length of the normal p to be $-C/\pm\sqrt{A^2+B^2}$ Since $\tan\theta = A/-B$ for the given line, we can obtain the expressions for $\cos\phi$ and $\sin\phi$. Substituting the expressions for $\cos\phi$, $\sin\phi$, and p in the equation $x\cos\phi+y\sin\phi-p=0$, gives

$$\frac{Ax+By+C}{\pm\sqrt{A^2+B^2}}=0\,.$$

II. Distance from a Point to a Line.

Let (x_0, y_0) be the point, and $Ax+By+C=0$ be the line. Translate the X-Y axes to the given point as a new origin, the equation of the given line becomes

(1) $$Ax'+By'+Ax_0+By_0+C=0\,.$$

Rotate the X'-Y' axes through the angle ϕ. Equation (1) becomes

(2) $$[A\cos\phi+B\sin\phi]x''-[A\sin\phi-B\cos\phi]y''+C=0\,.$$

In equation (2) setting $y''=0$, gives

(3) $$[A\cos\phi+B\sin\phi]x''=-(Ax_0+By_0+C)$$

Expressing $\cos\phi$ and $\sin\phi$ in terms of the values of $\sin\theta$ and $\cos\theta$, and substituting in (3), gives the absolute value of x'', which is

$$d=\frac{Ax_0+By_0+C}{\sqrt{A^2+B^2}}\,.$$

Sioux Falls College
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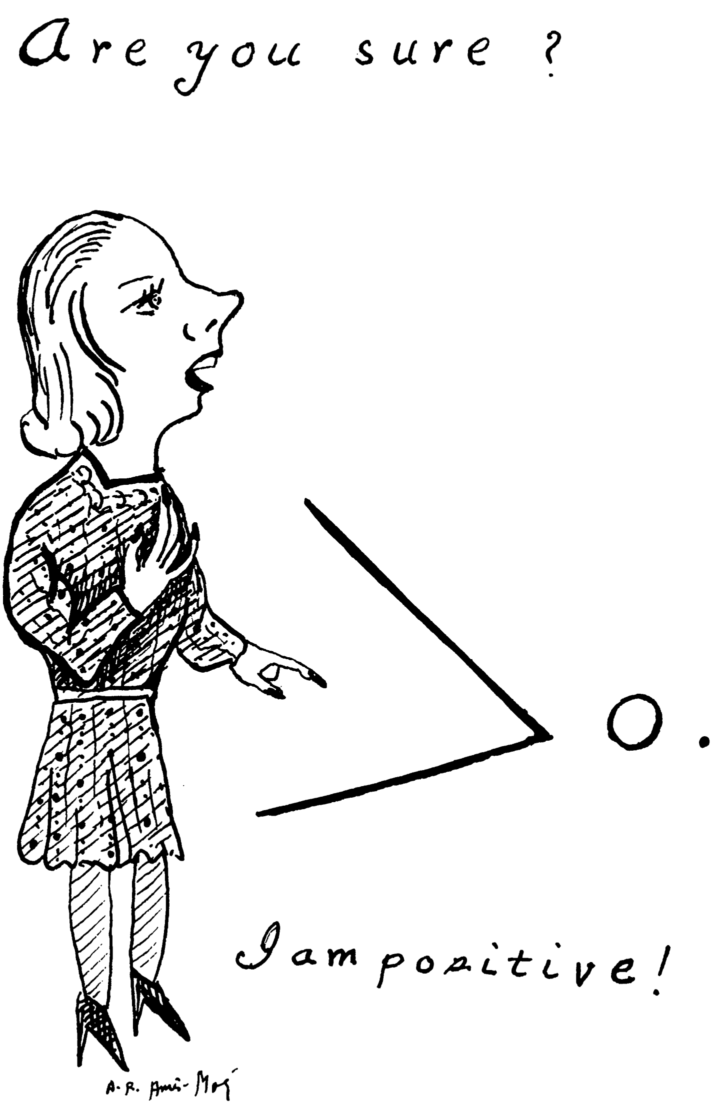
Are you sure ?
I am positive!

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.*

ON THE COEFFICIENTS OF $\dfrac{\cosh x}{\cos x}$

M. S. Krick

If

$$\frac{\cosh x}{\cos x} = \sum_{n=0}^{\infty} S_{2n} \frac{x^{2n}}{(2n)!},$$

then it is known that

$$\text{(I)} \qquad \sum_{r=0}^{n} (-1)^r \binom{2n}{2n-2r} S_{2n-2r} = 1,$$

where the S_{2i} are positive integers.[1] The values which appear below may be calculated from (I).

TABLE

$S_0 = 1$	$S_{10} = 126,752$
$S_2 = 2$	$S_{12} = 6,781,632$
$S_4 = 12$	$S_{14} = 500,231,552$
$S_6 = 152$	$S_{16} = 48,656,756,992$
$S_8 = 3472$	$S_{18} = 6,034,272,215,552$

$$S_{20} = 929,227,412,759,552$$

J. M. Gandhi conjectured from the table that

$$n(2n-1)S_{2n-2} \geqq S_{2n};$$

the proof follows.

Consider values of $n > 4$, and assume that for a given n

(II) $$k^2 S_{2k-2} < S_{2k}, \qquad (n-1 \geqq k \geqq 1).$$

Then, since for the values under consideration

$$\binom{2n}{2k} S_{2k} > \binom{2n}{2k-2} \frac{S_{2k}}{k^2},$$

it follows from (II) that

(III) $$\binom{2n}{2k} S_{2k} > \binom{2n}{2k-2} S_{2k-2}, \qquad (n-1 \geqq k \geqq 1).$$

If n is even, (I) may be rewritten

$$S_{2n} = \binom{2n}{2} S_{2n-2} + T_1 + T_{n-2} + \sum_{s=1}^{(\frac{n-4}{2})} (T_{2s} + T_{2s+1}),$$

where

(IV) $$T_i = (-1)^{n+i+1} \binom{2n}{2i} S_{2i}.$$

Now from (III) and (IV),

$$|T_{2s+1}| > |T_{2s}|, \qquad (\frac{n-4}{2} \geqq s \geqq 1)$$

$$T_{2s+1} > 0, \qquad (s \geqq 0).$$

Also it is easily verified that

$$\begin{aligned} n(n-1)S_{2n-2} &> \binom{2n}{2n-4} \frac{S_{2n-2}}{(n-1)^2} \\ &> \binom{2n}{2n-4} S_{2n-4} \quad [\text{by (II)}] \\ &= |T_{n-2}|. \end{aligned}$$

Hence

$$S_{2n} = n^2 S_{2n-2} + R_1,$$

where R_1 is some positive integer, and therefore

(V) $$n^2 S_{2n-2} < S_{2n}.$$

If n is odd, (I) may be rewritten

$$S_{2n} = n^2 S_{2n-2} + [n(n-1)S_{2n-2} + T_{n-2}] + 2 + \sum_{s=1}^{(\frac{n-3}{2})} (T_{2s-1} + T_{2s}) ,$$

and (V) follows as before. Since from the table

$$k^2 S_{2k-2} < S_{2k} , \qquad (5 \geqq k \geqq 1) ,$$

the validity of (V) for all n is established by induction.

For even $n > 2$, writing

$$S_{2n} = \binom{2n}{2} S_{2n-2} + \sum_{s=1}^{(\frac{n-2}{2})} (T_{2s-1} + T_{2s})$$

and observing that

$$|T_{2s}| > |T_{2s-1}| , \qquad T_{2s} < 0 \qquad (\frac{n-2}{2} \geqq s \geqq 1) ,$$

or, for odd $n > 3$, writing

$$S_{2n} = \binom{2n}{2} S_{2n-2} + (2 + T_1) + \sum_{s=1}^{(\frac{n-3}{2})} (T_{2s} + T_{2s+1})$$

and observing that

$$|T_{2s+1}| > |T_{2s}| , \qquad T_{2s+1} < 0 , \qquad |T_1| > 2 , \qquad (\frac{n-3}{2} \geqq s \geqq 0) ,$$

it is seen that

$$S_{2n} = n(2n-1)S_{2n-2} + R_2 ,$$

where R_2 is some negative integer, and therefore

$$\text{(VI)} \qquad n(2n-1)S_{2n-2} > S_{2n} , \qquad (n > 3) .$$

Finally, a combination of (V), (VI), and the table produces the general result

$$n^2 < \frac{S_{2n}}{S_{2n-2}} \begin{cases} < n(2n-1), & (n > 2) \\ = n(2n-1), & (n = 2) \\ > n(2n-1), & (n = 1) \end{cases} .$$

REFERENCE

(1) J. M. Gandhi, *The Coefficients of* $\cosh x/\cos x$ *and a Note on Carlitz's Coefficients of* $\sinh x/\sin x$, Mathematics Magazine, Vol. 31, No. 4, March-April, 1958.

Albright College
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Teachers of Methods Courses Please Note.

> "I think I probably owe a lot to Pollock, but some of them go back to the cave drawings, which is strange," Mr. Hale said. "Influences from many traditions are open to all of us, you know. What I have tried to do is to pierce the abstract barrier and come out on the other side, much the way a mathematician moves backward through the positive numbers and straight through zero into the negative realm. Lewis Carroll, of course, had this in mind when he sent Alice through the looking glass."

– The New Yorker, March 26, 1960.

Even this artist recognizes that zero **is** a number.

C. W. T.

A CORRECTION TO "IDEALS OF SQUARE SUMMABLE POWER SERIES" BY JAMES ROVNYAK*

James Rovnyak

Formula (5) in the proof of Theorem 5 is not justified since the series $B(z)g(z) = \Sigma B_n g(z) z^n$ is not known to converge in the Hilbert space metric. The difficulty can be overcome by establishing the theorems in a slightly different order. On the basis of Theorems 1-4, we prove Theorems 7, 8, and 5. The parts of the proofs of these theorems which are correct need not be mentioned.

To prove the sufficiency of Theorem 7, let $B_1(z) = \Sigma a_n z^n$ and $B_2(z) = \Sigma b_n z^n$. By the proof of Theorem 3, the series

$$B_1(z) B_2(z) = \Sigma\, a_n B_2(z) z^n$$

$$B_2(z) B_1(z) = \Sigma\, b_n B_1(z) z^n$$

converge in the Hilbert space metric. This justifies the expansion

$$\begin{aligned} \|B_1(z)B_2(z)\|^2 &= \langle B_1(z)B_2(z),\ B_2(z)B_1(z)\rangle \\ &= \Sigma\, a_m \bar{b}_n \langle B_2(z)z^m,\ B_1(z)z^n\rangle \\ &= \Sigma\, a_n \bar{b}_n \langle B_2(z),\ B_1(z)\rangle \\ &= \langle B_1(z),\ B_2(z)\rangle \langle B_2(z),\ B_1(z)\rangle \\ &= |\langle B_1(z),\ B_2(z)\rangle|^2 \end{aligned}$$

since the hypothesis implies $\langle B_2(z)z^m,\ B_1(z)z^n\rangle = 0$ whenever $m \neq n$. By Theorem 3, however,

$$|\langle B_1(z),\ B_2(z)\rangle|^2 = \|B_1(z)B_2(z)\|^2 = \|B_1(z)\|^2 \|B_2(z)\|^2$$

and by Halmos' discussion of the Schwarz inequality, [2] p. 15, $B_2(z) = cB_1(z)$, where, certainly, $|c| = 1$.

We now complete the proof of Theorem 8. Let **M** be a non-zero closed

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* See the Mathematics Magazine, May-June issue, vol. 33, no. 5 (1960).

ideal of $\mathbf{C}(z)$, and let $B(z)$ be an element of $\mathbf{M}$ of norm 1 orthogonal to every series $zf(z)$ where $f(z)$ ranges in $\mathbf{M}$. By Theorem 2, $B(z)$, $B(z)z$, $B(z)z^2$, $\cdots$ is an orthonormal set in $\mathbf{C}(z)$. The present proof that $\mathbf{M}(B)$ is an ideal is correct, and $\mathbf{M}(B) \subset \mathbf{M}$. To see that $\mathbf{M}(B)$ is all of $\mathbf{M}$, let $\mathbf{N}$ be the orthogonal complement of $\mathbf{M}(B)$ in $\mathbf{M}$. We use the results on orthogonality given in [2], paragraph 12, chapter 1.

$\mathbf{N}$ is a closed subspace of $\mathbf{M}$. Actually, $\mathbf{N}$ is an ideal. For, by the identity

$$\langle zf(z),\, B(z)g(z)\rangle = \langle f(z),\, B(z)[g(z)-g(0)]/z\rangle + \langle zf(z),\, g(0)B(z)\rangle$$

we see that $zf(z)$ is in $\mathbf{N}$ whenever $f(z)$ is in $\mathbf{N}$: the first inner product vanishes by the choice of $f(z)$ since $[g(z)-g(0)]/z$ is in $\mathbf{C}(z)$, and the second vanishes by the choice of $B(z)$.

We claim that $\mathbf{N}$ is the zero ideal. For, otherwise, we can choose a $B_1(z)$ in $\mathbf{N}$ such that $B_1(z)$, $B_1(z)z$, $B_1(z)z^2$, $\cdots$ is an orthonormal set in $\mathbf{C}(z)$ (See the comment after Theorem 2). But then, by the nature of $\mathbf{N}$ and the definition of $B(z)$, $\langle B_1(z)z^m,\, B(z)z^n\rangle = 0$ whenever $m \neq n$, and hence by Theorem 7, $B_1(z) = cB(z)$ for a suitable constant c, contradicting the assumption that $\mathbf{N}$ is orthogonal to $\mathbf{M}(B)$. Therefore, $\mathbf{M} = \mathbf{M}(B)$ and Theorem 8 is established.

To prove the sufficiency of Theorem 5, let $\mathbf{M}$ be the set of all $g(z)$ in $\mathbf{C}(z)$ such that $\langle g(z)z^m,\, B(z)\rangle = 0$ for $m = 1, 2, 3, \cdots$. Clearly, $\mathbf{M}$ is a closed ideal of $\mathbf{C}(z)$. By Theorem 8, there exists a $B_1(z)$ in $\mathbf{C}(z)$ such that $B_1(z)$, $B_1(z)z$, $B_1(z)z^2$, $\cdots$ is an orthonormal set in $\mathbf{C}(z)$ and such that $\mathbf{M} = \mathbf{M}(B_1)$. In particular, by the proof of Theorem 8,

$$\langle B_1(z)z^m,\, B(z)\rangle = 0 \quad \text{for} \quad m = 1,\ 2,\ 3,\ \cdots$$

and

$$\langle B_1(z),\, B(z)z^m\rangle = 0 \quad \text{for} \quad m = 1,\ 2,\ 3,\ \cdots$$

Now by Theorem 7, $B_1(z) = cB(z)$ and the sufficiency follows from the relation $\mathbf{M} = \mathbf{M}(cB)$.

Easton, Penn.

BOOK REVIEWS

Theory and Solution of Ordinary Differential Equations. By Donald Greenspan. The MacMillan Company, New York, 1960, viii + 148 pages. $5.50.

This is an excellent text designed for a one semester course in ordinary differential equations and is written in a frank, clear, and lively manner. The author states at the outset that he is writing for students who are familiar with the elements of advanced calculus and, except for the second chapter (first order equations), makes effective use of the calculus throughout the book. In short the book is written in the refreshing spirit of modern mathematics and, unlike the usual run of textbooks, adopts the attitude that our students can catch up to Newton (and possibly go beyond him). There are nine chapters in the book. The first discusses basic concepts. The third and fourth deal with the second and Nth order linear equations and make effective use of the Wronskian. Chapter five starts with a crystallization of the metric space concepts (actually, the author sets up a flavor for this in the previous chapters) and presents two proofs to the fundamental existence theorem: the first is based on the classical Picard method and the second proof is obtained as a by-product of contraction mappings. The remaining chapters cover linear systems, special functions, approximate solutions, and a survey of well-selected topics. There are a large number of formal exercises throughout the book together with problems of a serious nature.

Pasquale Porcelli

By Constance Reid. *From Zero to Infinity*. Second Revised Edition, Crowell, 1960, 153 pages. $3.95. and *Introduction to Higher Mathematics*. Crowell, 1959, 180 pages. $3.50.

The revised edition of Mrs. Reid's first book includes a new chapter on infinite sets and, unfortunately, omits the previous edition's delightful ending wherein it was shown that there is no uninteresting number. However, *From Zero to Infinity* remains a pleasure to read and anyone interested in integers will find it an accurately informative and an enjoyably readable introduction to the history and achievements of the theory of numbers.

Introduction to Higher Mathematics is written in the same narrative style as *From Zero to Infinity*. It gives a brief view of various fields of mathematics, such as matrix theory, non-euclidean geometry, sentential calculus, and group theory. The topics are developed historically and quotations of eminent mathematicians are frequent and pertinent. The illustrative examples are simple and non-trivial, as also are the few simple exercises included in some of the chapters.

The author of these books has accomplished the difficult task of writing descriptive and historical mathematics without becoming trite or overenthusiastic, and yet she often manages to impart to the reader a feeling akin to excitement over a story of a long struggle to solve a problem or to develop a method. This reviewer happily recommends these books to the reader with a mathematical background who will enjoy them, and to the non-mathematically trained reader who will also learn from them.

J. M. C. Hamilton

Elementary Statistics. By Sidney F. Mack. Henry Holt and Co., New York, 1960, ix+198. $4.50.

This is a text for a one semester course in beginning statistical inference requiring college algebra or a good grounding in high school algebra. The author does not treat all the topics covered in many comparable texts, but those topics that are included are covered with care. The material on hypothesis testing deserves special mention in this regard. Most of the carefully stated theorems are restricted to large samples. Motivation by use of worked out examples is a device used successfully throughout the text. Numerous illustrations and problems are included.

The author states that one of the reasons for writing the text was to attract more students into taking a statistics course for mathematics credit, instead of the traditional algebra and trigonometry courses. The modern flavor of the text merits the consideration of teachers with similar desires.

B. K. Gold

Mathematics in Action. By O. G. Sutton. Harper and Brothers, 1960, xvi+236 pages. $1.45.

This book is one of the Harper Torchbooks and brings to the reader a book of fundamental importance at a nominal cost. The cost factor is emphasized because I believe this excellent book is of great value in broadening the vision of our younger college students studying in the sciences and mathematics.

The book is an attempt to show how the basic disciplines of mathematics form the foundation for the theories of modern physical science. Going beyond this, the author shows how mathematics, both pure and applied, has influenced the discoveries of physicists and provided the scientists with models of the universe as well as the language in which their theories could be expressed. The book does this in language and a style that brings the most abstract of modern physical theories to the lay reader and college student at a level which is quite intelligible to him.

The first part of the book introduces the reader to the basic mathematical disciplines – arithmetic through calculus to probability and statistics. The author's clear explanations here could well be emulated by many of our textbook writers. Such subjects as non-Euclidean geometries, differential equations, the complex plane, and the Laplacian are covered in some 67 pages. What the reader gets from such an abbreviated treatment will depend upon his own mathematical background. Yet, the author does a remarkable job of presenting these ideas clearly and stripped of all unnecessary details.

The remainder of the book discusses the application to various fields of physical science of the mathematical disciplines which the early part of the book has developed. To cover all the major fields of physics would have required volumes, so some selection is necessary. The author chose those fields in which he was engaged in research. They form a varied, interesting, and quite representative set of scientific fields. They include: ballistics, wave mechanics, aerodynamics, and meteorology. These fields provide the opportunity to examine the classical Newtonian methods and contrast them with more recent mathematical methods. Of particular interest are his treatment of probability and statistics and the mathematics of weather forecasting.

I would recommend this book very highly for all secondary school teachers of science and mathematics. I have already suggested that my calculus students obtain it as a supplement to our text. And for those lay readers who are not afraid to face an occasional equation with some strange symbols, this book will provide the clearest explanation of mathematics in action that he is likely to find.

R. E. Horton

Modern High School Physics: A Recommended Course of Study. Second Edition, 1959, Bureau of Publications, Teachers College, Columbia University, vi + 70 pages. $1.50.

Members of the Scientific Manpower Project of Teachers College, Columbia University, are writing a series of monographs directed to improvement of science education. *Modern High School Physics* is one of five such monographs available; five more are in preparation.

This fine monograph should prove to be of great value to the conscientious physics teacher who is aware of the fact that almost the total content of the usual beginning physics course is nineteenth century physics, while the greater area of twentieth century development and discovery is at best treated as an afterthought during the final week or two of the semester. The teacher desiring to modernize his course can no longer just add the new to the classical, nor can he present both in the traditional manner; the enormously enlarged content of modern physics makes these procedures impossible. The authors of this monograph attempt to solve this dilemma and, in their attempt, give valuable suggestions to teachers desiring to offer more than a superficial survey of modern physics.

Over fifty pages of this book are devoted to the course content and the suggested outline of the high school physics course. Emphasizing the understanding of concepts, theories and principles, the course is presented in eight areas: Foundations of Mechanics, Wave Motion, Heat Energy, the Nature and Propagation of Light, Electricity, Magnetism and Electronics, Nuclear Energy, Relativity, and Recent Advances in Physics. Each of these topics is discussed and outlined in detail, the discussions presenting historical background and unifying ideas and interrelations among the eight areas. The recommended course does not attempt the impossible inclusion of all of present-day physics, but rather presents a well organized selection designed to give the future scientist or layman an understanding of important ideas and methods of modern physics.

The course can be used in its entirety, or as a guide in modernizing existing courses. A reader of this monograph will most likely be left with the resolve to learn as his students learn and to keep his course from becoming static during an era of sudden discovery and change.

J. M. C. Hamilton

Differential and Integral Calculus. By James R. F. Kent. Houghton Mifflin Company, Boston, 1960, xv + 511 pages. $6.75.

Modern calculus texts can be grouped into two classes: those which integrate calculus with algebra, analytic geometry and trigonometry; and those which treat differential then integral calculus as distinct subjects. Calculus texts can also be regarded as taking a position somewhere on a continuum ranging from extreme rigor of treatment to almost no rigor at all. This book is a member of the class of texts that deals with calculus as a separate subject. Therefore, it requires as prerequisites the subjects of algebra, trigonometry, and analytic geometry.

It is more difficult to properly place this book on the rigor continuum mentioned above. The author approaches the limit concept through an appeal to the student's intuition rather than by the more formal epsilon-delta method. However, he is more careful than many authors writing at this level to distinguish among right hand limits, left hand limits and the limit at a point. A further examination of Professor Kent's treatment of the Fundamental Theorem of Integral Calculus, infinite series, and functions of two or more variables would lead this reviewer to place this text at about the average level of rigor for a beginning calculus text.

The author mentions the various notations for the derivative but does not introduce the dy/dx notation until after defining the concept of the differential. He prefers to use the notation $D_x y$ throughout the book, a symbolism which seems to be gaining favor among the textbook writers.

The problem sets in the text are adequate and well organized. In the early part of the book one notes problems using terms from economics and business. However, the problems in later chapters are taken from the usual engineering, physical science, and mathematical sources.

The diagrams and figures are well drawn. A little greater use of italics and bold face type might have been used to emphasize important formulas and theorems. This reviewer would recommend this text as a good one for use in a standard approach to the college calculus course.

R. E. Horton

BOOKS RECEIVED FOR REVIEW

Brief Course in Analytics (Third edition). By M. A. Hill, Jr. and J. B. Linker. Holt, Rinehart, and Winston, Inc., New York, 1960, viii + 232 pages. $3.90.

Differential Equations (Second edition). By Ralph Palmer Agnew. McGraw-Hill Book Company, New York, 1960, vii + 485 pages. $7.50.

Complex Variables and Applications (Second edition). By Ruel V. Churchill. McGraw-Hill Book Company, New York, 1960, ix + 297 pages. $6.25.

Special Functions. By Earl D. Rainville. The Macmillan Company, New York, 1960, vi + 365 pages. $11.75.

Symbolic Logic. By Clarence I. Lewis and Cooper H. Langford. Dover Publications, Inc., New York, 1960, iv + 518 pages. $2.00.

Theory of Probability. By William Burnside. Dover Publications, Inc., New York, 1960, vi + 106 pages. $1.00.

Theory of Differential Equations. By Andrew R. Forsyth. Dover Publications, Inc., New York, 1960, xiii + 344 pages, x + 534 pages, xx + 596 pages. $15.00. Bound in three volumes.

Fundamental Principles of Mathematics. By John T. Moore. Rinehart and Company, Inc., New York, 1960, xv + 630 pages. $7.00.

Orbit Theory. Proceedings of Symposia in Applied Mathematics, Volume IX. American Mathematical Society, Providence, 1959, v + 196 pages.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India Ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

418. *Proposed by Maxey Brooke, Sweeney, Texas.*

The Pharaoh of Egypt, falling on hard times, decided to replenish the royal treasury by selling the Great Pyramid at one obol per cubic yard. The Emperor of Ethiopia, who had always wanted a pyramid, sent his royal surveyor to check on the measurements.

Now a yard was the length of the ruler's arm, and the Emperor had a longer arm than did the Pharaoh, so the volume of the pyramid as determined by the Ethiopian and Egyptian surveyors differed. There resulted much diplomatic wrangling over price.

It was finally decided to submit the question to the King of Babylon and abide by his decision. The King looked into the question and found that the Emperor's arm was as much longer than his own as the Pharaoh's was shorter. He reasoned that since the Babylonian yard was the average of the Ethiopian and Egyptian yards, the volume of the pyramid using his measure would be the average of the volumes as determined by the Egyptian and Ethiopian surveyors.

It is reputed that the Pharaoh came into an inheritance and no longer needed money so the pyramid was never sold.

Nevertheless, which ruler did the King's decision favor?

419. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Determine the path in a vertical plane such that when a particle moved, under gravity, with an initial velocity v_0 from a point of the path, the particle maintained a constant speed along the path. Assume no friction.

420. *Proposed by L. Carlitz, Duke University.*

Let

$$Q_{r,s} = \frac{(rs)!}{r!(s!)^r}.$$

Show that

$$Q_{r,ps} \equiv Q_{r,s} \pmod{p}$$

where p is a prime.

421. *Proposed by James Churchyard, Avondale, Arizona.*

Given a straight line segment ABC such that $AB \neq BC$. What is the locus of points P such that angle APB is equal to angle BPC?

422. *Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts.*

Solve the differential equation

$$\{x(1-\lambda)D^2 + (x\phi' + 1)D + x\phi'' + \phi'\}y = 0 ,$$

where λ is a constant and ϕ is a given function of x.

423. *Proposed by D. L. Silverman, University of Maryland.*

Prove that for real numbers a, b, and c,

$$|a-b| + |a+b-2c+|a-b|| < a+b$$

if and only if

$$|c-b| + |c+b-2a+|c-b|| < c+b .$$

424. *Proposed by E. R. Frankel, U. S. Department of Health, Education and Welfare.*

The general term of a polynomial series is

$$u_r = \binom{a+dr}{n} .$$

Show that

$$\Delta^n u_0 = d^n .$$

SOLUTIONS

Late Solutions

390, 392, 393, 394, 395, 396. *Melvin Hochster, Stuyvesant High School, New York.*

Errata

On page 300, Vol. 33, No. 5, May-June 1960, problem *384* should read *394*.

On page 226, Vol. 33, No. 4, March-April 1960 problem *369* in the last equation on the page and the first three equations on page 227 each occurrence of the symbol "sin" should be replaced by "csc".

On page 109, Vol. 33, No. 2, Nov.-Dec. 1959, problem *393* was originally submitted by Mr. Klamkin as a "Quickie".

On page 237, Vol. 33, No. 4, March-April 1960 in F 17 the name of *D. E. Whitford*, who jointly submitted it, was omitted.

A Diophantine Equation

397. [January 1960] *Proposed by Leo Moser, University of Alberta.*

Show that the Diophantine equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}+\frac{1}{x_1\cdot x_2\cdot x_3\cdots x_n}=1$$

has at least one solution for every n.

Solution by Charles F. Pinzka, University of Cincinnati.

A general solution is given by the recursive formula

$$x_r = x_1x_2\cdots x_{r-1}+1\ ,\quad 1<r\leqq n\ ,\quad r_1=2\ .$$

We note that the equation is satisfied for $n = 1$. Assuming that the set $(x_1, x_2, \cdots, x_r)$ satisfies the equation for $n = r$, then, for $n = r+1$ we have

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_{r+1}}+\frac{1}{x_1x_2\cdots x_{r+1}} = 1-\frac{1}{x_1\cdots x_r}+\frac{1}{x_{r+1}}+\frac{1}{x_1\cdots x_{r+1}}$$

$$= 1+\frac{1}{x_{r+1}}-\frac{x_{r+1}-1}{x_1\cdots x_{r+1}} = 1\ ,$$

completing the induction.

The solution is unique (except for permutation of the x_i) for $n = 1, 2$. For each $n > 2$ there exists an infinite set of solutions.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Maxey Brooke, Sweeny, Texas; R. G.. Buschman, Oregon State College; L. Carlitz, Duke University; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Harry M. Gehman, University of Buffalo; M. S. Klamkin, AVCO, Wilmington, Massachusetts; A. J. Kokar, School of Mines, Adelaide, Australia; Arne Pleijel, Trollhattan, Sweden; Robert E. Shafer, University of California Radiation Laboratory; William Squire, Southwest Research Institute, San Antonio, Texas; J. S. Vigder, Defence Research Board, Ottawa, Canada; Chih-yi Wang, University of Minnesota; Dale Woods, Northeast Missouri State College; and the proposer.

Simultaneous Quadratics

398. [January 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Determine the roots of the equations

$$x^2+y_1x+y_2 = 0$$

$$y^2+x_1y+x_2 = 0$$

where the coefficients (real numbers) in one equation are the roots of the other.

Solution by Harry M. Gehman, University of Buffalo.

The relations between roots and coefficients give these four equations:

$$x_1+x_2 = -y_1$$

$$x_1x_2 = y_2$$

$$y_1+y_2 = -x_1$$

$$y_1y_2 = x_2$$

From the first and third equations, $x_2 = y_2$.

Case I. If $x_2 = y_2 = 0$, then $x_1 = -y_1 = a$, where a is arbitrary, the equations are

$$x^2-ax = 0 \quad \text{and} \quad x^2+ax = 0$$

whose roots are a, 0 and $-a$, 0 respectively.

Case II. If $x_2 = y_2 \neq 0$, then $x_1 = y_1 = 1$, and $x_2 = y_2 = -2$. Both equations become

$$x^2+x-2 = 0$$

whose roots are 1, -2. Note that there is no need for the condition that the coefficients be real.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Sidney Kravitz, Dover, New Jersey; A. J. Kokar, School of Mines, Adelaide, Australia; Rostyslaw Lewyckyj, University of Toronto; Ernest E. Moyers, University of Mississippi; F. D. Parker, University of Alaska; Charles F. Pinzka, University of Cincinnati; Arne Pleijel, Trollhattan, Sweden; Robert E. Shafer, University of California Radiation Laboratory; C. M. Sidlo, Framingham, Massachusetts; William Squire, Southwestern Research Institute, San Antonio, Texas; Harvey Walden, Rensselaer Polytechnic Institute; Chih-yi Wang, University of Minnesota; Dale Woods, Northeastern Missouri State College; and the proposer.

A Cevian Triangle

399. [January 1960] *Proposed by Nathan Altshiller Court, University of Oklahoma.*

Prove that the feet of the perpendiculars dropped upon the sides of a triangle from their respective Simson poles form a cevian triangle. That is, the lines joining those points to the respectively opposite vertices are concurrent.

Solution by Sister M. Stephanie, Georgian Court College, New Jersey. Since the Simson pole of side BC of a triangle is the point diametrically opposite vertex A, the lines joining the feet of the perpendiculars to their respective vertices are isotomic conjugates of the altitudes. Such lines must be concurrent in the isotomically conjugate point of the orthocenter.

Further, (and this result is believed to be new), it can be shown by the use of complex coordinates that such a point is given by

$$z = \frac{3s_1s_3 + s_1^2s_2 - 4s_2^2}{s_1s_2 - 9s_3}.$$

The point lies on a line joining the centroid of the triangle with the symmedian point K. If we designate the point by P, then $\frac{PG}{GK} = \frac{2}{1}$, for (since $K = \frac{2s_2^2 - 6s_1s_3}{s_1s_2 - 9s_3}$),

$$\frac{\dfrac{2(2s_2^2 - 6s_1s_3)}{s_1s_2 - 9s_3} + \dfrac{3s_1s_3 + s_1^2s_2 - 4s_2^2}{s_1s_2 - 9s_3}}{3} = \frac{s_1}{3}.$$

Also solved by D. Moody Bailey, Princeton, West Virginia; J.W. Clawson, Collegeville, Pennsylvania; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; and the proposer.

A Triangle Construction

400. [January 1960] *Proposed by W. B. Carver, Cornell University.*

Given a point, a circle, and any curve in a plane. Construct an equilateral triangle having a vertex on each of them.

I. *Solution by R. G. Buschman, Oregon State College.*

The construction is not always possible as is shown by being given the point $P: (0, 0)$, the circle $C: x^2 + y^2 = 1$, and the line $L: x = 2$, since $\overline{PC} = 1$ and $\overline{PL} \geqq 2$. Thus a triangle with a vertex on each cannot be equilateral.

Given a circle and a point we can choose a coordinate system such that $P: (a, 0)$ and $C: x^2 + y^2 = 1$. Consider a curve $F: f(x, y) = 0$ which is to be the locus of points such that $\overline{PF}^2 = \overline{PC}^2 = \overline{CF}^2$. After some manipulation and the elimination of the coordinates of the point on the circle

$$f(x, y) = y^2[x^2+y^2-1-2ax]^2+[(x-a)(x^2+y^2-1)-2ax^2]^2-(2ay)^2 .$$

Thus for the equilateral triangle to be constructable the given curve in the plane must have at least one intersection with the curve F.

II. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Let A, (b), (c) be the given point, circle and curve. If, then, ABC is the solution, the vertex C on (c) is obtained from the vertex B on (b) by a rotation about A with the angles 60 and -60 degrees. Hence the unknown vertices on (c) are obtained by intersecting (c) with new positions of (b) after such rotations. The constructions of the triangles are then immediate.

Also solved by Harry M. Gehman, University of Buffalo; Rostyslaw J. Lewyckyj, University of Toronto; Harvey Walden (partial solution); and the proposer.

A Fibonacci Series

401. [January 1960] *Proposed by John M. Howell, Los Angeles City College.*

Given a sequence of numbers related by $F(n) = aF(n-1)+bF(n-2)$, $F(0) = c$ and $F(1) = d$, where $n = 0, 1, 2, \cdots$ and a, b, c, and d are any real numbers. Find a general form for $F(n)$.

I. *Solution by J.L. Brown, Jr., Ordinance Research Laboratory, Pennsylvania State University.*

If $a = 0$, then $F(2n) = b^n c$ and $F(2n+1) = b^n d$ for $n \geqq 1$; similarly if $b = 0$, then $F(n) = a^{n-1}d$ for $n \geqq 1$. Assume, therefore, that neither a nor b is zero in the following:

Let t_1 and t_2 denote the roots of the equation $t^2-at-b = 0$.

Case 1: $(t_1 \neq t_2)$:

Then $F(n) = \propto t_1^n+\beta t_2^n$ for $n \geqq 0$. Using the initial conditions to evaluate $\propto$ and β,

$$F(n) = \frac{ct_2-d}{t_2-t_1}\, t_1^n+\frac{d-ct_1}{t_2-t_1}\, t_2^n .$$

Case 2: $(t_1 = t_2)$:

In this case,

$$F(n) = \propto t_1^n+\beta n t_2^n \quad \text{for} \quad n \geqq 0 ,$$

or,

$$F(n) = ct_1^n+\left(\frac{2d}{a}-c\right)nt_1^n ,$$

making use of the initial conditions.

NOTE: This sequence has been considered previously by H. Siebeck,

J. Neuberg, and N. Traverso, among others. For detailed references and related work on generalizations of the Fibonacci series, see Chapter 17, Vol. 1 of L. E. Dickson's "History of the Theory of Numbers," Chelsea, 1952.

II. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

We have successively

$$F(0)=c \qquad\qquad = c$$

$$F(1)=d \qquad\qquad = d$$

$$F(2)=aF(1)+bF(0) \qquad\qquad = \binom{1}{0}ad+\binom{0}{0}bc$$

$$F(3)=aF(2)+bF(1)$$

$$=(a^2+b)F(1)+abF(0) \qquad\qquad = \Big[\binom{2}{0}a^2+\binom{1}{1}b\Big]d+\Big[\binom{1}{0}ab\;\; c$$

$$F(4)=aF(3)+bF(2)$$

$$=(a^3+2ab)F(1)+(a^2b+b^2)F(0)=\Big[\binom{3}{0}a^3+\binom{2}{1}ab\Big]d+\Big[\binom{2}{0}a^2b+\binom{1}{1}b^2\Big]c$$

$$\cdots \qquad\qquad \cdots$$

and in general

$$F(n)=\Big[\binom{n-1}{0}a^{n-1}+\binom{n-2}{1}a^{n-3}b+\binom{n-3}{2}a^{n-5}b^2+\cdots\Big]d$$

$$+\Big[\binom{n-2}{0}a^{n-2}+\binom{n-3}{1}a^{n-4}b+\binom{n-4}{2}a^{n-6}b^2+\cdots\Big]bc$$

which may be proved by induction.

III. *Alternate solution by Huseyin Demir.*

Writing the relation for $n=2, \cdots, n$ we have a system of equations in the unknowns $F(2), \cdots, F(n)$:

$$F(n)-aF(n-1)-bF(n-2) = 0$$

$$F(n-1)-aF(n-2)-bF(n-3) = 0$$

$$\cdots$$

$$F(4)-aF(3)-bF(2)=0$$

$$F(3)-aF(2)=bF(1)$$

$$F(2)=aF(1)-bF(0)$$

The determinant of the system being 1 we have

$$F(n) = \begin{vmatrix} 0 & -a & -b & 0 & \cdots & 0 \\ 0 & 1 & -a & -b & 0 & 0 \\ & 0 & 1 & -a & -b & \\ & & \cdots & & & \\ 0 & 0 & & 1 & -a & -b \\ bd & 0 & & 0 & 1 & -a \\ ad+bc & 0 & \cdots & & 0 & 1 \end{vmatrix}_{n-1}$$

where the index denotes the order of the determinant.

Expanding it with respect to the first column and arranging, we have the final result

$$F(n) = (bc+ad)\begin{vmatrix} a & b & 0 & \cdots & 0 \\ -b & a & b & & \vdots \\ 0 & -b & a & b & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & -b & a & b \\ 0 & \cdots & 0 & -b & a \end{vmatrix}_{n-2} + bd\begin{vmatrix} a & b & 0 & \cdots & 0 \\ -b & a & b & & \vdots \\ 0 & -b & a & b & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & -b & a & b \\ 0 & \cdots & 0 & -b & a \end{vmatrix}_{n-3}$$

IV. *Alternate solution by Huseyin Demir.*

Writing the given relation

$$F(n) = aF(n-1) + bF(n-2)$$

in the form

$$\frac{F(n)}{F(n-1)} = a + \frac{b}{F(n-1)/F(n-2)}$$

and letting $u_n = F(n)/F(n-1)$ we have successively

$$u_n = a + b/u_{n-1}$$

$$= a + \frac{b}{a + b/u_{n-2}}$$

$$\cdots$$

$$u_n = a + \cfrac{b}{a + \cfrac{b}{a + \ddots \, a + \cfrac{b}{a + d/c}}}$$

Multiplying member to member the relations

$$F(n) = u_n F(n-1)$$

$$\cdot \quad \cdot \quad \cdot$$

$$F(2) = u_2 F(1)$$

$$F(1) = u_1 F(0)$$

we have for the general form for $F(n)$:

$$F(n) = u_1 u_2 \cdots u_n \cdot c$$

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; R. G. Buschman, University of Oregon; F.D. Parker, University of Alaska; Charles F. Pinzka, University of Cincinnati; Chih-yi Wang, University of Minnesota; and the proposer.

A Binomial Sum

402. [January 1960] *Proposed by Chih-yi Wang, University of Minnesota.*

I. Show that if $|x| < 1$,

$$F(x) = \sum_{n=k}^{\infty} \binom{n}{k} x^n = x^k (1-x)^{k-1} .$$

II. Prove the same equality without using the binomial coefficients of negative arguments.

Solution by Dmitri Thoro, San Jose State College.

Repeated differentiation of the expression

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n , \quad |x| < 1$$

yields

$$\frac{k!}{(1-x)^k} = \sum_{n=k}^{\infty} n(n-1)(n-2) \cdots (n-k+1) x^{n-k} .$$

Multiplication by $x^k/k!$ gives the desired result:

$$\sum_{n=k}^{\infty} \binom{n}{k} x^k = \frac{x^k}{(1-x)^{k+1}} , \quad |x| < 1 .$$

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; Rostyslaw J. Lewyckyj, University of Toronto; Charles F. Pinzka, University *of Cincinnati; Arne Pleijel, Trollhattan, Sweden; William Squire, Southwest Research Institute, San Antonio, Texas; and the proposer.*

Concurrent Lines

403. [January 1960] *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Given four lines, a_i ($i = 1, 2, 3, 4$), determining a four space. Prove that if a_i meets a_{i+1} (mod 4), then all the given lines concur.

Solution by R. G. Buschman, Oregon State College.

Write the equations of the lines in the form

$$a_i : X_i = X_i^* + A_i t\,, \quad i = 1, 2, 3, 4,$$

where X_i, X_i^*, A_i are vectors. Since a_i meets a_{i+1}, then $X_i = X_{i+1}$ for some t, say t_i. (We reduce subscripts mod 4 throughout the discussion.) Thus

$$X_i^* - X_{i+1}^* = (A_i - A_{i+1})t_i\,,$$

so that on adding these four equations, we obtain

$$0 = \sum_{i=1}^{4} (A_i - A_{i+1})t_i + \sum_{i=1}^{4} A_i(t_i - t_{i+1})\,.$$

Thus we have a system of 4 homogeneous equations in $(t_i - t_{i+1})$ from the components of the vectors with the coefficient matrix composed of the components of the vectors A_i. If the rank of this matrix were less than 4, then the direction vectors of the lines would be dependent and the 4 lines would not determine a 4-space. Thus the rank is 4 and the only solution gives $t_1 = t_2 = t_3 = t_4$, i.e., the lines have a point in common.

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 268. Show that one cannot inscribe a regular polygon of more than four sides in an ellipse (with unequal axes). [*Submitted by M. S. Klamkin*]

Q 269. If a, b, and c are positive numbers, give a geometrical interpretation for the inequality

$$2[a^2b^2 + b^2c^2 + c^2a^2] \geqq a^4 + b^4 + c^4 \ .$$

[*Submitted by M. S. Klamkin*]

ANSWERS

A 268. Assume that it can be done. Then there would exist a circle intersecting the ellipse in more than four points which is impossible.

A 269.

$$4a^2b^2 \geqq (a^2 + b^2 + c^2)^2$$

or

$$[(a+b)^2 - c^2][c^2 - (a-b)^2] \geqq 0 \ .$$

Now assume $a \geqq b \geqq c$, then $c \geqq a - b$.
Consequently, a, b, and c form a triangle.

BERTRAND CURVES ASSOCIATED WITH A PAIR OF CURVES

John F. Burke

1. INTRODUCTION. A theorem due to C. Bioche (1) states that if there exists a one-to-one correspondence between the points of the curves C_1 and C_2 such that at corresponding points P_1 on C_1 and P_2 on C_2

(a) The curvature κ_1 of C_1 is constant

(b) The torsion τ_2 of C_2 is constant

(c) The unit tangent vector T_1 of C_1 is parallel to the unit tangent vector T_2 of C_2,

then the curve C generated by the point P that divides the segment P_1P_2 in the ratio $m:1$ is a Bertrand curve.

It is the purpose of this paper to prove that if condition (c) is modified so that the binormals B_1 and B_2 at P_1 and P_2 are parallel, then the curve C is a Bertrand curve; and if condition (c) is modified so that the tangent T_1 at P_1 is parallel to the binormal B_2 at P_2 then the curve C is again a Bertrand curve.

Gibbs methods and notation of vector analysis will be used in this paper.

2. THEOREM 1. If condition (c) of Bioche's theorem is modified so that at corresponding points P_1 and P_2 the binormals B_1 and B_2 are parallel, then the curve C is a Bertrand curve.

Proof: Since $B_1 = B_2$ then $\tau_1 N_1 = \tau_2 N_2 \dfrac{ds_2}{ds_1}$ where N_1 and N_2 are the unit normal vector of C_1 and C_2 at P_1 and P_2 respectively, and ds_1 and ds_2 are the respective elements of arc length.

Hence $N_1 = N_2$ and $T_1 = N_1 \times B_1 = N_2 \times B_2 = T_2$ therefore T_1 is parallel to T_2 so that by Bioche's theorems, C is a Bertrand curve.

THEOREM 2. If condition (c) of Bioche's theorem is modified so that at corresponding points P_1 and P_2 the tangent T_1 at P_1 is parallel to the binormal B_2 at P_2, then the curve C is a Bertrand curve.

Proof: Since

(1) $$T_1 = B_2$$

it follows that

(2) $$\kappa_1 N_1 = -\tau_2 N_2 \frac{ds_2}{ds_1}.$$

Hence N_1 is parallel to N_2 and since N_1 and N_2 are unit vectors,

(3) $$N_1 = N_2$$

and

(4) $$\frac{ds_2}{ds_1} = -\frac{\kappa_1}{\tau_2}$$

also

(5) $$B_1 = -T_2$$

since $B_1 = T_1 \times N_1 = B_2 \times N_2 = -T_2$.

Let R, R_1, R_2 be the coordinate vectors at the points P, P_1, P_2 on the curves C, C_1, C_2 respectively. Then

(6) $$R = mR_1 + (1-m)R_2$$

Differentiating (6) one gets, using (4)

(7) $$\begin{aligned} T &= mT_1 \frac{ds_1}{ds} + (1-m)T_2 \frac{ds_2}{ds} \\ &= [mT_1 - \frac{\kappa_1}{\tau_2}(1-m)T_2] \frac{ds_1}{ds} \\ &= m_1 T_1 + m_2 T_2 \end{aligned}$$

where

$$m\frac{ds_1}{ds} = \frac{m\tau_2}{\sqrt{(\tau_2 m)^2 + [\kappa_1(1-m)]^2}} = m_1$$

$$(1-m)\frac{ds_2}{ds} = \frac{-(1-m)\kappa_1}{\sqrt{(\tau_2 m)^2 + [\kappa_1(1-m)]^2}} = m_2$$

m_1 and m_2 constants.

Differentiating (7) one gets,

(8) $$\kappa N = \kappa_1 m_1 \frac{ds_1}{ds} N_1 + \kappa_2 m_2 \frac{ds_2}{ds} N_2 .$$

Hence

(9) $$N = N_1 = N_2$$

and

(10) $$\kappa = \kappa_1 m_1 \frac{ds_1}{ds} + \kappa_2 m_2 \frac{ds_2}{ds} .$$

Using (7) and (9), one finds that

(11) $$B = m_1 B_1 + m_2 B_2 .$$

Differentiating (11) one gets,

(12) $$\tau N = \tau_1 m_1 \frac{ds_1}{ds} N_1 + \tau_2 m_2 \frac{ds_2}{ds} N_2 .$$

Hence

$$\tau = \tau_1 m_1 \frac{ds_1}{ds} + \tau_2 m_2 \frac{ds_2}{ds} . \tag{13}$$

Using (4) and (5), one gets

$$\frac{ds_2}{ds_1} = \frac{\tau_1}{\kappa_2} = -\frac{\kappa_1}{\tau_2} \tag{14}$$

and

$$\frac{\tau_1}{\kappa_1} = -\frac{\kappa_2}{\tau_2} . \tag{15}$$

Let

$$M_1 = m_1 \frac{ds_1}{ds} \quad , \quad M_2 = m_2 \frac{ds_2}{ds}$$

then using (10) and (13), one gets

$$\frac{\kappa}{M_2 \tau_2} + \frac{\tau}{M_1 \kappa_1} = \frac{\kappa_1}{M_2 \tau_2} - \frac{\tau_2}{M_1 \kappa_1} \qquad \text{(a constant)}$$

and this is the intrinsic equation of a Bertrand curve.

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REFERENCE

C. Bioche, Bull. Soc. Math. France 17, 1888-89, p.p. 112.

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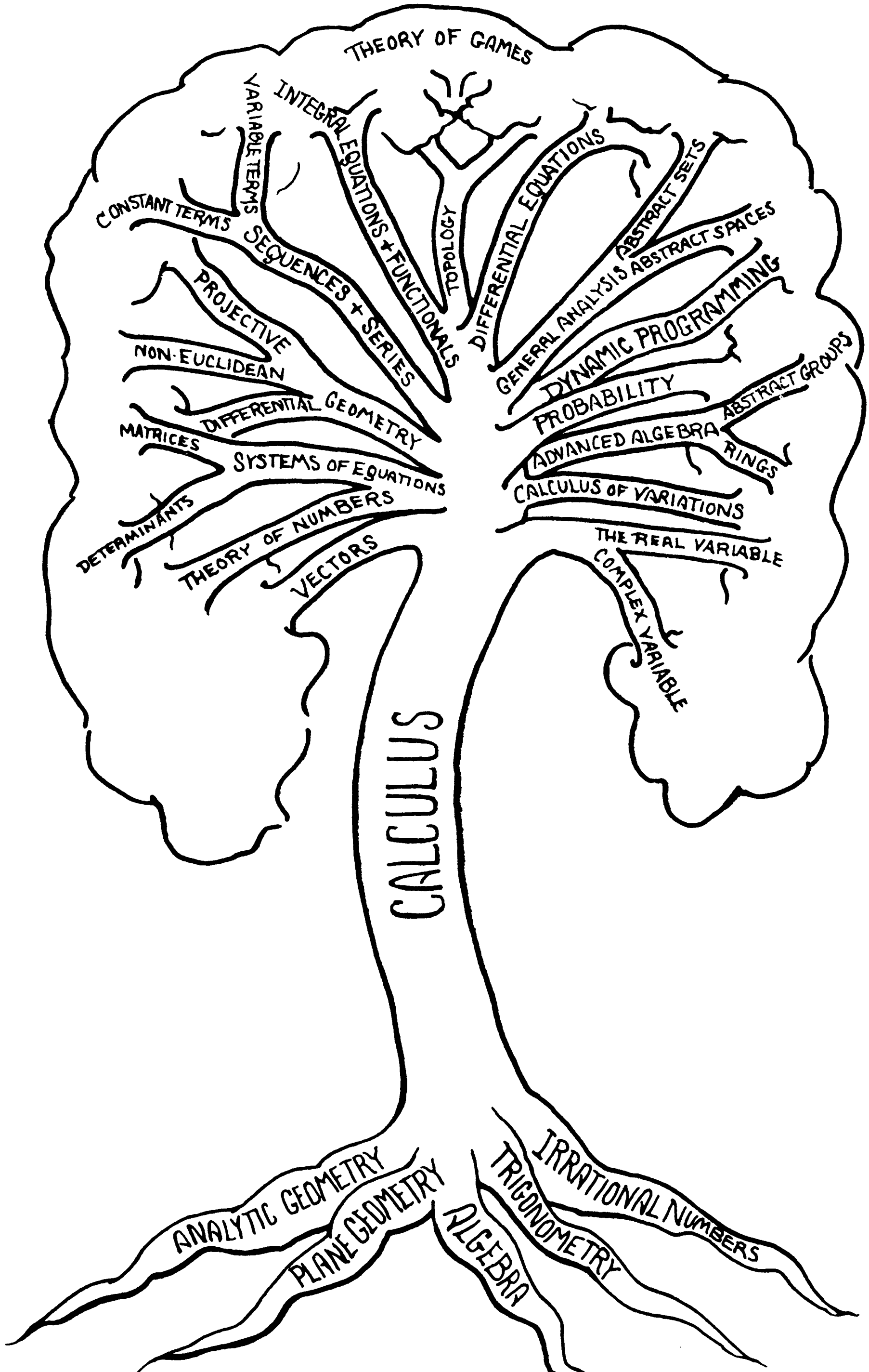

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